Methodol Comput Appl Probab (2012) 14:285–297
DOI 10.1007/s11009-010-9203-x

Random Motion on Simple Graphs

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Received: 21 June 2010 / Revised: 17 October 2010 /
Accepted: 20 October 2010 / Published online: 10 November 2010
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Abstract Consider a stochastic process that lives on n-semiaxes joined at the origin. On each ray it behaves as one dimensional Brownian Motion and at the origin it chooses a ray uniformly at random (Kirchhoff condition). The principal results are the computation of the exit probabilities and certain other probabilistic quantities regarding exit and occupation times.

Keywords Brownian motion · Kirchhoff condition · Exit probabilities · Exit times · Occupation times · Arc-sine law

AMS 2000 Subject Classifications 60J60 · 60J65 · 60J70

1 Introduction

Suppose we have a system of S semiaxes with a common origin and a particle moving randomly on S. Here we are interested: (a) In the time when the particle reaches at a certain point of S and (b) how long does the particle spend in a specific part of S, say in one of the rays that constitute S. Possible applications include spread of toxic particles in a system of channels or vessels or propagation of information in networks (see, e.g., Deng and Li 2009).

The mathematical model is the following: Let S be the set consisting of n semiaxes $S_1, \ldots, S_n$, $n \geq 2$, with a common origin 0 and X_t the Brownian motion process on S, namely the diffusion process on S whose infinitesimal generator L is

$$Lu = \frac{1}{2}u'', \tag{1}$$

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where

$$u = (u_1, ..., u_n),$$

together with the continuity conditions (a total of $n-1$ equations),

$$u_1(0) = \cdots = u_n(0) \tag{2}$$

and the so-called "Kirchhoff condition"

$$u'_1(0) + \cdots + u'_n(0) = 0. \tag{3}$$

This is a Walsh's-type Brownian motion (see Barlow et al. 1989).

It is well-known that L defines a (unique) self-adjoint operator on the space

$$L_2(S) = \bigoplus_{j=1}^{n} L_2(S_j) \simeq \bigoplus_{j=1}^{n} L_2(0, \infty).$$

The process X_t does a standard Brownian motion on each of the semiaxes and, when it hits 0, it continues its motion on the j-th semiaxis, $1 \le j \le n$, with probability $1/n$, (this is the probabilistic meaning of Eq. 3, see, e.g., Freidlin and Wentzell 1993). For notational clarity it is helpful to use the coordinate x_j, $0 \le x_j < \infty$, for the semiaxis S_j, $1 \le j \le n$. Notice that, if $u = (u_1, ..., u_n)$ is a function on S, then its j-th component, u_j, is a function on S_j, hence $u_j = u_j(x_j)$.

We have computed the transition density of X_t (see Fig. 1):

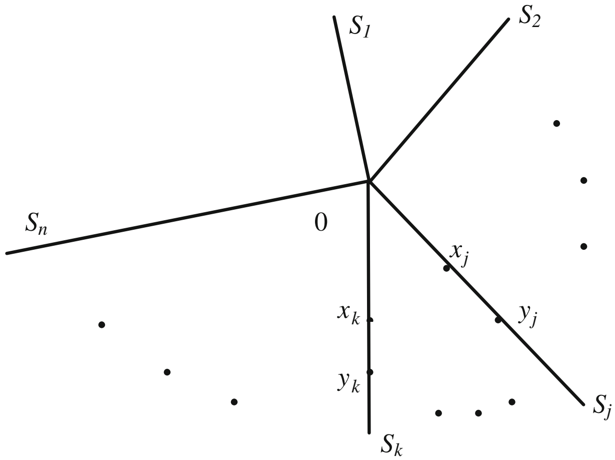
$$p(t, x_k, y_j) = \frac{2}{n\sqrt{2\pi t}} e^{-\frac{(x_k+y_j)^2}{2t}},$$

if $k \neq j$, and

$$p(t, x_k, y_k) = \frac{1}{\sqrt{2\pi t}} \left[e^{-\frac{(x_k-y_k)^2}{2t}} - \frac{n-2}{n} e^{-\frac{(x_k+y_k)^2}{2t}} \right].$$

We want to study certain issues regarding exit (or hitting) times and occupational times of X_t. Our results extend certain classical results for the standard Brownian

Fig. 1 The graph S



motion on $\mathbb{R}$ (e.g. the continuous gambler's ruin problem—see, e.g., Chung and Zambrini 2001) which actually corresponds to the case $n = 2$ ($x_1 = x$, $x_2 = -x$, where $x \geq 0$). Finally, we want to mention that Brownian motion-diffusion models are often used in environmental research (see for example Hristopulos 2003; James et al. 2005; Mtundu and Koch 1987).

2 Exit Times and Exit Probabilities (Explicit Calculations)

On each semiaxis S_j, $1 \leq j \leq n$, consider the point $b_j > 0$. These points define the (bounded) subset of S

$$S^b = \bigcup_{j=1}^{n} \{x_j : 0 \leq x_j < b_j\}$$

(thus S^b consists of n line segments of lengths $b_1, ..., b_n$, with a common initial point, namely 0). We assume that $X_0 \in S^b$ and we denote by T the exit time from S^b, i.e. the smallest time such that $X_t = b_j$, for some $j = 1, ..., n$. We also introduce the events

$$B_j = \left\{X_T = b_j\right\}. \tag{4}$$

If $X_0 = x_j$, we denote the associated probability measure by P^{x_j} and the expectation by E^{x_j}.

Let us now consider the following boundary value problem for $u = (u_1, ..., u_n)$

$$\frac{1}{2}u_j'' - \lambda u_j = 0, \qquad j = 1, ..., n, \tag{5}$$

where λ is a complex parameter and u satisfies Eqs. 2 and 3 namely

$$u_1(0) = \cdots = u_n(0), \tag{6}$$

$$u_1'(0) + \cdots + u_n'(0) = 0, \tag{7}$$

together with the boundary conditions

$$u_1\left(b_1\right) = 1, \tag{8}$$

and

$$u_j\left(b_j\right) = 0, \qquad j \neq 1. \tag{9}$$

The solution u of the above problem has the Feynman–Kac representation (see, e.g., Freidlin 1885 or Karatzas and Shreve 1991)

$$u_j\left(x_j\right) = u_j\left(x_j; \lambda\right) = E^{x_j}\left[e^{-\lambda T}\mathbf{1}_{B_1}\right] \tag{10}$$

as long as

$$\Re\{-\lambda\} < \lambda_1,$$

where λ_1 is the smallest eigenvalue of L acting on S^b with Dirichlet (i.e. 0) boundary conditions at $x_j = b_j$, $j = 1, ..., n$. In particular Eq. 10 is valid for all $\lambda \geq 0$. It is straightforward to check that λ_1 is the smallest positive zero of

$$F(\lambda) = \cot\left(\sqrt{2\lambda}b_1\right) + \cdots + \cot\left(\sqrt{2\lambda}b_n\right).$$

Set $b_M = \max\{b_1, ..., b_n\}$. Since $F(0+) = +\infty$ and $F((\pi^2/2b_M^2)-) = -\infty$, it follows that

$$0 < \lambda_1 < \frac{\pi^2}{2b_M^2}.$$

Let us calculate the solution u of the problem Eqs. 5–9. First assume $\lambda \neq 0$. To satisfy Eqs. 5, 8, and 9 we must take

$$u_1(x_1) = A_1 \sinh\left[\sqrt{2\lambda}(x_1 - b_1)\right] + \cosh\left[\sqrt{2\lambda}(x_1 - b_1)\right] \tag{11}$$

and

$$u_j(x_j) = A_j \sinh\left[\sqrt{2\lambda}(x_j - b_j)\right], \qquad 2 \leq j \leq n. \tag{12}$$

To determine the constants $A_1, ..., A_n$ we have to use Eqs. 6 and 7. From Eq. 6 we get

$$A_1 \sinh\left(\sqrt{2\lambda}b_1\right) - \cosh\left(\sqrt{2\lambda}b_1\right) = A_2 \sinh\left(\sqrt{2\lambda}b_2\right) = \cdots = A_n \sinh\left(\sqrt{2\lambda}b_n\right),$$

hence

$$A_j = \frac{A_1 \sinh\left(\sqrt{2\lambda}b_1\right) - \cosh\left(\sqrt{2\lambda}b_1\right)}{\sinh\left(\sqrt{2\lambda}b_j\right)}, \qquad 2 \leq j \leq n. \tag{13}$$

By Eq. 7 we have

$$A_1 \cosh\left(\sqrt{2\lambda}b_1\right) - \sinh\left(\sqrt{2\lambda}b_1\right) + A_2 \cosh\left(\sqrt{2\lambda}b_2\right) + \cdots + A_n \cosh\left(\sqrt{2\lambda}b_n\right) = 0. \tag{14}$$

Using Eq. 13 in Eq. 14 yields

$$\begin{aligned} A_1 \cosh\left(\sqrt{2\lambda}b_1\right) &+ A_1 \sinh\left(\sqrt{2\lambda}b_1\right) \coth\left(\sqrt{2\lambda}b_2\right) + \cdots \\ &+ A_1 \sinh\left(\sqrt{2\lambda}b_1\right) \coth\left(\sqrt{2\lambda}b_n\right) \\ &= \sinh\left(\sqrt{2\lambda}b_1\right) + \cosh\left(\sqrt{2\lambda}b_1\right) \coth\left(\sqrt{2\lambda}b_2\right) + \cdots \\ &\quad + \cosh\left(\sqrt{2\lambda}b_1\right) \coth\left(\sqrt{2\lambda}b_n\right) \end{aligned}$$

or

$$A_1 \sinh\left(\sqrt{2\lambda}b_1\right)\left[\coth\left(\sqrt{2\lambda}b_1\right)+\coth\left(\sqrt{2\lambda}b_2\right)+\cdots+\coth\left(\sqrt{2\lambda}b_n\right)\right]$$
$$=\sinh\left(\sqrt{2\lambda}b_1\right)\left[1+\coth\left(\sqrt{2\lambda}b_1\right)\coth\left(\sqrt{2\lambda}b_2\right)+\cdots\right.$$
$$\left.+\coth\left(\sqrt{2\lambda}b_1\right)\coth\left(\sqrt{2\lambda}b_n\right)\right].$$

Therefore

$$A_1=\frac{1+\coth\left(\sqrt{2\lambda}b_1\right)\sum_{k=2}^{n}\coth\left(\sqrt{2\lambda}b_k\right)}{\sum_{k=1}^{n}\coth\left(\sqrt{2\lambda}b_k\right)} \tag{15}$$

and hence Eq. 13 becomes

$$A_j=-\frac{1}{\sinh\left(\sqrt{2\lambda}b_j\right)\sinh\left(\sqrt{2\lambda}b_1\right)\sum_{k=1}^{n}\coth\left(\sqrt{2\lambda}b_k\right)}, \tag{16}$$

for $2 \le j \le n$.

Let us also analyze the somehow exceptional case $\lambda = 0$. In this case the Eq. 5 becomes

$$u_j'' = 0, \qquad j = 1, ..., n, \tag{17}$$

and the boundary conditions are again Eqs. 6, 7, 8, and 9. By formula 10 we can see immediately that the solution u has the probabilistic interpretation

$$u_j\left(x_j\right) = u_j\left(x_j; 0\right) = E^{x_j}\left[\mathbf{1}_{B_1}\right] = P^{x_j}\left[B_1\right] = P^{x_j}\left\{X_T = b_1\right\}. \tag{18}$$

To satisfy Eqs. 17, 8, and 9 we must take

$$u_1\left(x_1\right) = A_1(x_1 - b_1) + 1$$

and

$$u_j\left(x_j\right) = A_j(x_j - b_j), \qquad 2 \le j \le n.$$

To determine the constants $A_1, ..., A_n$ we have to use Eqs. 6 and 7. From Eq. 6 we get

$$A_1 b_1 - 1 = A_2 b_2 = \cdots = A_n b_n,$$

hence

$$A_j = \frac{A_1 b_1 - 1}{b_j}, \qquad 2 \le j \le n. \tag{19}$$

On the other hand, from Eq. 7 we get

$$A_1 + A_2 + \cdots + A_n = 0. \tag{20}$$

Using Eq. 19 in Eq. 20 yields

$$A_1 + \frac{A_1 b_1}{b_2} + \cdots + \frac{A_1 b_1}{b_n} = \frac{1}{b_2} + \cdots + \frac{1}{b_n}$$

or

$$A_1 b_1 \left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}\right) = \frac{1}{b_2} + \cdots + \frac{1}{b_n}.$$

Therefore

$$A_1 = \frac{\sum_{k=2}^{n}(1/b_k)}{b_1 \sum_{k=1}^{n}(1/b_k)} \tag{21}$$

and hence Eq. 19 becomes

$$A_j = -\frac{1}{b_j b_1 \sum_{k=1}^{n}(1/b_k)}, \tag{22}$$

for $2 \leq j \leq n$.

We summarize the above results in the following theorem:

Theorem 1 *For $\lambda > 0$ (or more generally for $\lambda \neq 0$, $\Re\{-\lambda\} < \lambda_1$, where λ_1 is the smallest eigenvalue of L acting on S^b with Dirichlet boundary conditions at $x_j = b_j$, $j = 1, ..., n$) we have*

$$E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{B_j}\right] = \frac{1}{\sinh\left(\sqrt{2\lambda}b_i\right)\sinh\left(\sqrt{2\lambda}b_j\right)\sum_{k=1}^{n}\coth\left(\sqrt{2\lambda}b_k\right)}\sinh\left[\sqrt{2\lambda}(b_i - x_i)\right],$$

if $i \neq j$. Also

$$E^{x_j}\left[e^{-\lambda T}\mathbf{1}_{B_j}\right]$$
$$= \cosh\left[\sqrt{2\lambda}(b_j - x_j)\right] + \left[\frac{1}{\sinh\left(\sqrt{2\lambda}b_j\right)\sum_{k=1}^{n}\coth\left(\sqrt{2\lambda}b_k\right)} - \cosh\left(\sqrt{2\lambda}b_j\right)\right]$$
$$\times \frac{\sinh\left[\sqrt{2\lambda}(b_j - x_j)\right]}{\sinh\left(\sqrt{2\lambda}b_j\right)}.$$

If $\lambda = 0$, the above formulas become

$$E^{x_i}\left[\mathbf{1}_{B_j}\right] = P^{x_i}\left[B_j\right] = P^{x_i}\left\{X_T = b_j\right\} = \frac{1}{b_j \sum_{k=1}^{n}(1/b_k)}\left(1 - \frac{x_i}{b_i}\right),$$

if $i \neq j$. Finally

$$E^{x_j}\left[\mathbf{1}_{B_j}\right] = P^{x_j}\left[B_j\right] = P^{x_j}\left\{X_T = b_j\right\} = \frac{x_j}{b_j} + \frac{1}{b_j \sum_{k=1}^{n}(1/b_k)}\left(1 - \frac{x_j}{b_j}\right).$$

Remark If we set $x_i = 0$ (or $x_j = 0$) in the above formulas, we obtain

$$E^{0}\left[e^{-\lambda T}\mathbf{1}_{B_j}\right] = \frac{1}{\sinh\left(\sqrt{2\lambda}b_j\right)\sum_{k=1}^{n}\coth\left(\sqrt{2\lambda}b_k\right)}. \tag{23}$$

In particular ($\lambda = 0$)

$$E^{0}\left[\mathbf{1}_{B_j}\right] = P^{0}\left[B_j\right] = P^{0}\left\{X_T = b_j\right\} = \frac{1}{b_j \sum_{k=1}^{n}(1/b_k)}. \tag{24}$$

Next we consider the problem

$$\frac{1}{2}U_j'' - \lambda U_j = 0, \qquad j = 1, ..., n, \tag{25}$$

$$U_1(0) = \cdots = U_n(0), \tag{26}$$

$$U_1'(0) + \cdots + U_n'(0) = 0, \tag{27}$$

$$U_i(b_i) = 1, \qquad i = 1, ..., n. \tag{28}$$

Here, the meaning of $U(x_j)$ is $E^{x_j}\left[e^{-\lambda T}\right]$, the moment generating function of T, when the exit axis is not specified. It follows (e.g., see again Freidlin 1885 or Karatzas and Shreve 1991) that

$$U(x_i) = U(x_i; \lambda) = E^{x_i}\left[e^{-\lambda T}\right]$$

and

$$U(0; \lambda) = E^0\left[e^{-\lambda T}\right].$$

Notice that

$$U(x_i) = E^{x_i}\left[e^{-\lambda T}\right] = \sum_{j=1}^{n} E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{B_j}\right], \tag{29}$$

where each $E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{B_j}\right]$ (including $i = j$) is given by Theorem 1. Setting $x_i = 0$ and using Eq. 23 we obtain the following corollary:

Corollary 1

$$E^0\left[e^{-\lambda T}\right] = \frac{1}{\sum_{k=1}^{n} \coth\left(\sqrt{2\lambda}b_k\right)} \sum_{j=1}^{n} \frac{1}{\sinh\left(\sqrt{2\lambda}b_j\right)}. \tag{30}$$

Having Eq. 30 we can use a probabilistic approach to compute $E^{x_i}\left[e^{-\lambda T}\right]$ (instead of trying to solve the problem Eqs. 25–28):

Assume $X_0 = x_i$. Let T_i be the (first) exit time from the line segment $0 < x_i < b_i$, so that $X(T_i) = 0$, or $X(T_i) = b_i$, in which case $T_i = T$. Then we have

$$E^{x_i}\left[e^{-\lambda T}\right] = E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{X(T_i)=b_i\}}\right] + E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{X(T_i)=0\}}\right],$$

or equivalently

$$E^{x_i}\left[e^{-\lambda T}\right] = E^{x_i}\left[e^{-\lambda T_i}\mathbf{1}_{\{X(T_i)=b_i\}}\right] + E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{X(T_i)=0\}}\right]. \tag{31}$$

The first term of the right-hand side of Eq. 31 can be computed by applying Theorem 1 with $n = 2$, $b_1 = b_i$, and $b_2 = 0$ (see also Chung and Zambrini 2001 or Karatzas and Shreve 1991). The result is

$$E^{x_i}\left[e^{-\lambda T_i}\mathbf{1}_{\{X(T_i)=b_i\}}\right] = \frac{\sinh\left(\sqrt{2\lambda}x_i\right)}{\sinh\left(\sqrt{2\lambda}b_i\right)}.$$

Next we show how to compute the second of the right-hand side of Eq. 31 with the help of the strong Markov property of X_t and Corollary 1. Let T_0 be the (first) time at which X_t hits 0. Then $\{X(T_i) = 0\} = \{T_0 < T\}$ and

$$E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{X(T_i)=0\}}\right] = E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{T_0<T\}}\right] = E^{x_i}\left\{E\left[e^{-\lambda T}\mathbf{1}_{\{T_0<T\}}\middle|\,\mathscr{F}_{T_0}\right]\right\},$$

hence

$$E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{X(T_i)=0\}}\right] = E^{x_i}\left\{e^{-\lambda T_0}\mathbf{1}_{\{T_0<T\}}E^0\left[e^{-\lambda T}\right]\right\} = E^0\left[e^{-\lambda T}\right]E^{x_i}\left[e^{-\lambda T_0}\mathbf{1}_{\{T_0<T\}}\right],$$

i.e.

$$E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{X(T_i)=0\}}\right] = E^0\left[e^{-\lambda T}\right]E^{x_i}\left\{e^{-\lambda T_i}\mathbf{1}_{\{X(T_i)=0\}}\right\}.$$

Thus, by using Corollary 1 (and Theorem 1) we obtain

$$E^{x_i}\left[e^{-\lambda T}\mathbf{1}_{\{X(T_i)=0\}}\right] = \frac{1}{\sum_{k=1}^{n}\coth\left(\sqrt{2\lambda}b_k\right)}\left[\sum_{j=1}^{n}\frac{1}{\sinh\left(\sqrt{2\lambda}b_j\right)}\right]\frac{\sinh\left[\sqrt{2\lambda}(b_i - x_i)\right]}{\sinh\left(\sqrt{2\lambda}b_i\right)}.$$

We summarize our result in the following theorem:

Theorem 2

$$E^{x_i}\left[e^{-\lambda T}\right] = \frac{\sinh\left(\sqrt{2\lambda}x_i\right)}{\sinh\left(\sqrt{2\lambda}b_i\right)} + \frac{1}{\sum_{k=1}^{n}\coth\left(\sqrt{2\lambda}b_k\right)}\left[\sum_{j=1}^{n}\frac{1}{\sinh\left(\sqrt{2\lambda}b_j\right)}\right]\frac{\sinh\left[\sqrt{2\lambda}(b_i - x_i)\right]}{\sinh\left(\sqrt{2\lambda}b_i\right)}. \quad (32)$$

Remark An interesting special case of the above theorem is when

$$b_1 = \cdots = b_n = b.$$

Then, formula 32 becomes

$$E^{x_i}\left[e^{-\lambda T}\right] = \frac{\sinh\left(\sqrt{2\lambda}x_i\right)}{\sinh\left(\sqrt{2\lambda}b\right)} + \frac{\sinh\left[\sqrt{2\lambda}(b - x_i)\right]}{\cosh\left(\sqrt{2\lambda}b\right)\sinh\left(\sqrt{2\lambda}b\right)}.$$

Notice that there is no dependence on n. In particular

$$E^0\left[e^{-\lambda T}\right] = \frac{1}{\cosh\left(\sqrt{2\lambda}b\right)}.$$

We finish the section with some more explicit formulas relating the exit time T and the events B_j. The solutions $u_j\left(x_j;\lambda\right)$, $j = 1, ..., n$, of Eqs. 5–9 are entire functions of λ (of order 1/2). If we expand them about $\lambda = 0$ as

$$u_j\left(x_j;\lambda\right) = u_j\left(x_j;0\right) + \lambda v_j\left(x_j\right) + O\left(\lambda^2\right),$$

then $v = (v_1, ..., v_n)$ satisfies the system

$$\frac{1}{2}v_j'' = -u_j\left(x_j; 0\right), \qquad j = 1, ..., n, \tag{33}$$

$$v_1(0) = \cdots = v_n(0), \tag{34}$$

$$v_1'(0) + \cdots + v_n'(0) = 0, \tag{35}$$

$$v_j\left(b_j\right) = 0, \qquad j = 1, ..., n. \tag{36}$$

By Theorem 1

$$u_j\left(x_j; 0\right) = \frac{1}{b_j \sum_{k=1}^{n}(1/b_k)}\left(1 - \frac{x_j}{b_j}\right), \tag{37}$$

if $j \neq 1$ and

$$u_1\left(x_1; 0\right) = \frac{x_1}{b_1} + \frac{1}{b_1 \sum_{k=1}^{n}(1/b_k)}\left(1 - \frac{x_1}{b_1}\right). \tag{38}$$

The solution v of the above problem has the probabilistic representation

$$v_j\left(x_j\right) = E^{x_j}\left[T\mathbf{1}_{B_j}\right] \tag{39}$$

since $u_j\left(x_j; \lambda\right) = E^{x_j}\left[e^{-\lambda T}\mathbf{1}_{B_j}\right]$ (this is Eq. 10) and

$$v_j\left(x_j\right) = \left.\frac{\partial}{\partial\lambda}u_j\left(x_j; \lambda\right)\right|_{\lambda=0}.$$

Let us rewrite Eq. 33 in the form

$$\frac{1}{2}v_j'' = \gamma_j\left(x_j - b_j\right) + \delta_j, \qquad j = 1, ..., n, \tag{40}$$

where, in view of Eqs. 37 and 38

$$\gamma_j = \frac{1}{b_j^2 \sum_{k=1}^{n}(1/b_k)}, \qquad \delta_j = 0, \tag{41}$$

for $j \neq 1$, while

$$\gamma_1 = \frac{1}{b_1^2 \sum_{k=1}^{n}(1/b_k)} - \frac{1}{b_1}, \qquad \delta_1 = -1. \tag{42}$$

Then a straightforward calculation gives that

$$v_j\left(x_j\right) = \frac{\gamma_j\left(x_j - b_j\right)^3}{3} + \delta_j\left(x_j - b_j\right)^2 + \varepsilon_j\left(x_j - b_j\right), \tag{43}$$

where

$$\varepsilon_j = \delta_j b_j - \frac{\gamma_j b_j^2}{3} + \frac{\sum_{k=1}^{n}\left(\delta_k b_k - (2/3)\gamma_k b_k^2\right)}{b_j \sum_{k=1}^{n}(1/b_k)}. \tag{44}$$

Finally if $V = (V_1, ..., V_n)$ satisfies the system

$$\frac{1}{2}V_j'' = -1, \qquad j = 1, ..., n, \tag{45}$$

$$V_1(0) = \cdots = V_n(0), \tag{46}$$

$$V_1'(0) + \cdots + V_n'(0) = 0, \tag{47}$$

$$V_j\left(b_j\right) = 0, \qquad j = 1, ..., n, \tag{48}$$

Then

$$V_j(x_j) = E^{x_j}[T].$$

By taking

$$\gamma_j = 0, \qquad \delta_j = -1$$

in Eqs. 43 and 44 we obtain the following corollary:

Corollary 2

$$E^{x_j}[T] = \left[b_j + \frac{\sum_{k=1}^n b_k}{b_j \sum_{k=1}^n (1/b_k)}\right]\left(b_j - x_j\right) - \left(b_j - x_j\right)^2,$$

or equivalently

$$E^{x_j}[T] = x_j\left(b_j - x_j\right) + \frac{\sum_{k=1}^n b_k}{\sum_{k=1}^n (1/b_k)}\left(1 - \frac{x_j}{b_j}\right).$$

In particular

$$E^0[T] = \frac{\sum_{k=1}^n b_k}{\sum_{k=1}^n (1/b_k)}.$$

3 A Generalization of the Arc-Sine Law

Let m be the Lebesgue measure of $\mathbb{R}^1$. We introduce the occupation times of S_j:

$$\Gamma_j(t) = m\left\{s \in [0, t] : X_s \in S_j\right\} = \int_0^t \mathbf{1}_{S_j}(X_s)\, ds. \tag{49}$$

Of course,

$$0 \leq \Gamma_j(t) \leq t \qquad \text{and} \qquad \sum_{j=1}^n \Gamma_j(t) = t. \tag{50}$$

For $n = 2$ the famous arcsine law of P. Lévy states that (see, e.g. Karatzas and Shreve 1991)

$$P^0\{\Gamma_1(t) \leq \alpha t\} = \frac{1}{\pi}\int_0^\alpha \frac{dr}{\sqrt{r(1-r)}} = \frac{2}{\pi}\arcsin(\alpha). \tag{51}$$

We want to extend the above formula for $n > 2$. Consider the problem

$$\frac{1}{2} w_j'' \left(x_j\right) - \alpha w_j \left(x_j\right) - \beta_j w_j \left(x_j\right) = -1, \qquad j = 1, ..., n-1, \tag{52}$$

$$\frac{1}{2} w_n'' \left(x_n\right) - \alpha w_n \left(x_n\right) = -1, \tag{53}$$

$$w_1(0) = \cdots = w_n(0), \tag{54}$$

$$w_1'(0) + \cdots + w_n'(0) = 0, \tag{55}$$

$$w_j \left(x_j\right) \text{ bounded on } S_j, \qquad j = 1, ..., n. \tag{56}$$

Here $\alpha, \beta_1, ..., \beta_{n-1}$ are parameters. The solution $w = (w_1, ..., w_n)$ of this problem has the Feynman–Kac representation (see, e.g., Freidlin 1885 or Karatzas and Shreve 1991)

$$w_j \left(x_j\right) = E^{x_j} \left[\int_0^\infty e^{-\alpha t} \exp \left(- \int_0^t \sum_{k=1}^{n-1} \beta_k \mathbf{1}_{S_k} \left(X_s\right) ds \right) dt \right] \tag{57}$$

(notice that, for a given value of X_s, at most one of the quantities $\mathbf{1}_{S_j} \left(X_s\right)$ does not vanish). In particular (see Eq. 49)

$$w_j \left(0\right) = \int_0^\infty e^{-\alpha t} E^0 \left[\exp \left(- \sum_{k=1}^{n-1} \beta_k \Gamma_k (t) \right) \right] dt, \tag{58}$$

independently of j due to Eq. 54.

Now the solution of Eqs. 52–56 is

$$w_j \left(x_j\right) = A_j e^{-x_j \sqrt{2(\alpha + \beta_j)}} + \frac{1}{\alpha + \beta_j}, \qquad j = 1, ..., n-1,$$

and

$$w_n \left(x_n\right) = A_n e^{-x_n \sqrt{2\alpha}} + \frac{1}{\alpha},$$

where, in view of Eqs. 54 and 55, the constants $A_1, ..., A_n$ must satisfy

$$A_1 + \frac{1}{\alpha + \beta_1} = \cdots = A_{n-1} + \frac{1}{\alpha + \beta_{n-1}} = A_n + \frac{1}{\alpha} = w_j(0)$$

and

$$A_1 \sqrt{\alpha + \beta_1} + \cdots + A_{n-1} \sqrt{\alpha + \beta_{n-1}} + A_n \sqrt{\alpha} = 0.$$

It follows that

$$w_j(0) = \left(\frac{1}{\sqrt{\alpha + \beta_1}} + \cdots + \frac{1}{\sqrt{\alpha + \beta_{n-1}}} + \frac{1}{\sqrt{\alpha}} \right) \frac{1}{\sqrt{\alpha + \beta_1} + \cdots + \sqrt{\alpha + \beta_{n-1}} + \sqrt{\alpha}}.$$

We summarize the above observations in the following proposition:

Proposition 1 *Let*

$$h\,(t;\,\beta_1,\,...,\,\beta_{n-1}) = E^0\left[\exp\left(-\sum_{k=1}^{n-1}\beta_k\Gamma_k(t)\right)\right], \qquad t \geq 0,$$

where $\Gamma_k(t)$, $k = 1, ..., n-1$, *are the occupation times of Eq. 49. If*

$$H\,(\alpha;\,\beta_1,\,...,\,\beta_{n-1}) = \int_0^{\infty} e^{-\alpha t} h\,(t;\,\beta_1,\,...,\,\beta_{n-1})\,dt$$

is the Laplace transform of h, then

$$H\,(\alpha;\,\beta_1,\,...,\,\beta_{n-1}) = \left(\frac{1}{\sqrt{\alpha+\beta_1}} + \cdots + \frac{1}{\sqrt{\alpha+\beta_{n-1}}} + \frac{1}{\sqrt{\alpha}}\right)\frac{1}{\sqrt{\alpha+\beta_1} + \cdots + \sqrt{\alpha+\beta_{n-1}} + \sqrt{\alpha}}.$$

Remark If we set $\beta_1 = \beta$ and $\beta_2 = \cdots = \beta_{n-1} = 0$, then the above function h becomes

$$h_1\,(t;\,\beta) := h\,(t;\,\beta,0,...,0) = E^0\left[\exp\left(-\beta\Gamma_1(t)\right)\right], \qquad t \geq 0.$$

Let

$$H_1\,(\alpha;\,\beta) = \int_0^{\infty} e^{-\alpha t} h_1\,(t;\,\beta)\,dt.$$

Then

$$H_1\,(\alpha;\,\beta) = \left(\frac{1}{\sqrt{\alpha+\beta}} + \frac{n-1}{\sqrt{\alpha}}\right)\frac{1}{\sqrt{\alpha+\beta} + (n-1)\sqrt{\alpha}}.$$

For $n = 2$ we obtain the standard quantity that appears in the arcsine law.

If $f(x;t)$ is the probability density function of $\Gamma_1(t)$, then, of course,

$$h_1\,(t;\,\beta) = E^0\left[\exp\left(-\beta\Gamma_1(t)\right)\right] = \int_0^{\infty} e^{-\beta x} f(x;t)dx.$$

Hence

$$\int_0^{\infty}\int_0^{\infty} e^{-\alpha t} e^{-\beta x} f(x;t)dxdt = \left(\frac{1}{\sqrt{\alpha+\beta}} + \frac{n-1}{\sqrt{\alpha}}\right)\frac{1}{\sqrt{\alpha+\beta} + (n-1)\sqrt{\alpha}}.$$

4 Conclusion and Discussion

In this paper we have discussed a stochastic process X_t that lives on a system of S semiaxes with a common origin. X_t does a standard Brownian Motion on each of the semiaxes and when it hits 0, it continues its motion on the j-th semiaxis $1 \leq j \leq n$, with probability $1/n$. We have computed explicitly some exit probabilities and the associated exit times. Our results extend certain classical results for the standard Brownian Motion on $\mathbb{R}$ (e.g. the continuous gambler's ruin problem). Furthermore, we have made an attempt to extend the arcsine law of P. Lévy to the occupation times $\Gamma_j(t)$ of our process X_t on each branch of S. We have computed explicitly the (double) Laplace transform of the density of $\Gamma_j(t)$. However the exact form of this density remains an open problem.

Acknowledgement The work was partially supported by a Π.E.B.E. 2008 Grant of the National Technical University of Athens.

References

Barlow MT, Pitman JW, Yor M (1989) On Walsh's Brownian motions. Sem. prob. XXIII. In: Lecture notes in mathematics, vol 1372. Springer, Berlin, pp 275–293

Chung KL, Zambrini JC (2001) Introduction to random time and quantum randomness. In: Monographs of the Portuguese mathematical society. McGraw-Hill, Lisboa

Deng D, Li Q (2009) Communication in naturally mobile sensor networks. Wireless algorithms, systems, and applications. Lect Notes Comput Sci 5682:295–304

Freidlin MI (1885) Functional integration and partial differential equations. Princeton University Press, Princeton

Freidlin MI, Wentzell AD (1993) Diffusion processes on graphs and the averaging principle. Ann Probab 21(4):2215–2245

Hristopulos DT (2003) Permissibility of fractal exponents and models of band-limited two-point functions for fGn and fBm random fields. Stoch Environ Res Risk Assess 17:191–216

James CS, Bilezikjian KT, Chysikopoulos VC (2005) Contaminant transport in a fracture with spatially variable aperture in the presence of monodisperse and polydisperse colloids. Stoch Environ Res Risk Assess 19:266–279

Karatzas I, Shreve SE (1991) Brownian motion and stochastic calculus, 2nd edn. Graduate texts in mathematics, vol 113. Springer, New York

Mtundu ND, Koch RW (1987) A stochastic differential equation approach to soil moisture. Stoch Hydrol Hydraul 1:101–116