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A two-unit parallel system supported by $(n-2)$ standbys with general and non-identical lifetimes

EFFIE PAPAGEORGIOU†* and GEORGE KOKOLAKIS†

This paper examines a functioning policy of a parallel system. We assume availability of n non-identical, non-repairable units for replacement or support. Two units start their operation simultaneously at times $S_1 = S_2 = 0$, *and any one of them is replaced instantaneously upon its failure by one of the* $(n-2)$ *standby units at random starting times* S_i $(i = 3, \ldots, n)$. *Thus, with probability one, the system is functioning with two units up till the failure of the* $(n-1)$*th unit. Unit lifetimes* T_i $(i = 1, \ldots, n)$ *have a general joint distribution function* $F(\mathbf{t})$. *The system has to operate for a fixed period of time, c, and it stops functioning when all available units fail before c. The probability that the system is functioning for the required period of time c depends on the distribution of the unit lifetimes. The reliability of the system is evaluated by recursive relations. Independent unit lifetimes are considered as special cases.*

1. Introduction

Repairable two-unit redundant systems have attracted the attention of several researchers working in the field of reliability theory. Many workers, including Murari and Goel (1984), Gupta and Goel (1989), Goel *et al.* (1992), and Gupta and Chaudhary (1992), have investigated the two-unit standby system models assuming that, on the failure of one unit, it is replaced by the standby unit instantaneously. Gupta and Kishan (1999) consider situations of a two-unit system where the standby unit does not operate instantaneously, but a fixed preparation time is required to put standby and repaired units into operation. Several workers including Gopalan *et al.* (1986), Gupta and Goel (1990), and Murari and Maruthachalam (1981) have analysed two-unit system models under a variety of assumptions using the regenerative point technique. In all these system models, it is assumed that the lifetimes are uncorrelated random variables. Gupta *et al.* (1999) have investigated a two non-identical parallel system with correlated lifetimes and independently distributed repair times. In these models, the repair time may be fixed or random and with or without administrative delay.

Other workers studied the age replacement policy, i.e. the unit is replaced at its failure or at pre-assigned age T, whichever comes first, assuming that a spare for replacement is always available. Nakagawa and Osaki (1974) extended the age-replacement model by considering that the spare for replacement may not always be available in practice. Osaki and Yamada (1976) discussed an age replacement policy with random delivery time and costs for stock and system downtime. Jhang (2001) considered an extension of the model studied by Osaki and Yamada (1976) by introducing age-dependent minimal repair and general random repair cost.

In all these parallel system models, it is assumed that the system failure occurs only when both units stop functioning, but it is not necessary for both units to work simultaneously. The structure of these system models allows the operation of only one unit during the repair time of the other. The main difference of the model we examine in this paper is that we have available a fixed number of n non-repairable and non-identical units with non-independent lifetimes with general distributions. Two units operate simultaneously until, at least, the entrance of the last standby unit. The two-unit parallel system is up as long as there is at least one working unit. Our main result is the determination of the reliability of the system by using probabilistic analysis.

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We have to emphasize here that our basic results refer to non-independent unit lifetimes with general distributions, unlike most earlier results which refer to specific unit lifetime distributions.

In Kokolakis *et al.* (1996), a similar n-unit, non-repairable redundant parallel system is considered. The units start at prescheduled times and up to n units may be working simultaneously. The system reliability is derived for independent lifetimes with general distributions, and its optimization with respect to the unit prescheduled starting times is investigated. Comparison with this policy is presented in Section 5.1.

1.1. *Nomenclature*

U_i	the ith unit introduced into the system,
T_i	lifetime of ith unit,
$f_i(\cdot)$, $F_i(\cdot)$	PDF and CDF of the ith unit lifetime,
$f(\mathbf{t}), F(\mathbf{t})$	joint PDF and CDF of unit lifetimes,
$R(\cdot)$	reliability function,
S_i	starting time of operation of the ith unit,
n	number of available and non-repairable units,
N	number of units used,
T	system lifetime,
c	fixed required period of system operation,
$\mathcal{S}$	the event $\{T \geq c\}$.

2. System description

The problem of the successful control of a process by a non-identical unit parallel system is considered in this paper.

Here, there is available a fixed number of n non-repairable and non-identical units. The unit lifetimes $T_i\,(i = 1, \dots, n)$ are random and not necessarily independently distributed with a general joint distribution, F. The process is considered to have a fixed duration, c. The control of the process is considered successful provided that at least one of the units is in operation during the required period of time, and the process information required for the control is transferred instantaneously to a new entering unit from a working one. The process is initially controlled by two units, and the remaining $(n-2)$ units are standbys. The two initial units start their operation simultaneously, i.e. at times $S_1 = S_2 = 0$, and one of these is replaced by a new one upon its failure, i.e. the third unit starts its operation at time $S_3 = \min\{T_1, T_2\}$. Similarly, the fourth unit starts its operation upon the failure time of one of the two working units, i.e. at time $S_4 = \min\{\max\{T_1, T_2\}, S_3 + T_3\}$. In general, the ith unit starts its operation at time $S_i = \min\{\max\{T_1, T_2, S_3 + T_3, \dots, S_{i-2} + T_{i-2}\}, S_{i-1} + T_{i-1}\}$ and it works in parallel with that already introduced and not failed before S_i. Thus, the times S_i $(i = 3, 4, \dots, n)$ are random, and the process is simultaneously controlled by two working units until, at least, the entrance of the last available unit. The system stops functioning when all units fail. The probability of the successful control of the process until its completion time, c, depends entirely on the distributions of unit-lifetimes.

In many realistic situations, the above policy describes the control of a process by a parallel system. As an example, we can consider a satellite communication system which consists of two satellites in orbit and $(n-2)$ standbys. If at least one of the two satellites is functioning in orbit, the system is considered a success, whereas if no one satellite functions in orbit, the system is considered a failure. The replenishment policy to keep the system operating successfully for a required time period, c, is as follows: Two satellites launch simultaneously and function in orbit. If one of them stops operating successfully in orbit, a new one launches and replaces it. So until the failure of the $(n-1)$ available, there are always two satellites operating in orbit. We assume that the time required to place a satellite in orbit after launch is negligible. The required process information is transferred instantaneously to the new entering satellite from the working one. It is important to know the probability that the system is operating successfully for the required time period, c.

3. Model analysis

Let T be the system lifetime and T_i $(i = 1, \dots, n)$ be the lifetimes of the units U_i with joint distribution, F. Let also $\mathcal{S}$ be the event of the successful control of the process during the required period of time c, i.e. $\mathcal{S} = \{T \geq c\}$.

From the description of the policy above, it follows that the system is functioning non-stop until, at least, the entrance of the unit U_n at time $S_n < c$. Thus, with the notation introduced, the problem considered here is as follows:

Problem: Evaluate the probability of the event $\mathcal{S}$ that the control of the process will be successful until time $t = c$. That is, we have to evaluate the probability $P[\mathcal{S}] = P[T \geq c]$.

To clarify the problem, we evaluate the above probability for independent lifetimes and with $n = 2$, 3, and 4 units.

3.1. *Two independent units*

For $n = 2$, we have:

$$S_1 = S_2 = 0.$$

The system lifetime $T = \max\{T_1, T_2\}$ and

$$P[\mathcal{S}] = P[T \geq c] = P[\max\{T_1, T_2\} \geq c] \\ = 1 - P[\max\{T_1, T_2\} < c],$$

and due to independence

$$P[\mathcal{S}] = 1 - P[T_1 < c]P[T_2 < c] \\ = 1 - (1 - P[T_1 \geq c])(1 - P[T_2 \geq c]).$$

Thus, with R_i, the reliability function of the ith unit, we have

$$P[\mathcal{S}] = 1 - \{1 - R_1(c)\}\{1 - R_2(c)\}. \tag{1}$$

Figure 1 refers to the case where both units fail before c.

3.2. *Three independent units*

For $n = 3$, we have:

$$S_1 = S_2 = 0 \quad \text{and} \quad S_3 = \min\{T_1, T_2\}.$$

Thus, $T = \max\{T_1, T_2, T_3 + S_3\} = \max\{\max\{T_1, T_2\}, T_3 + S_3\}$ and

$$P[\mathcal{S}] = P[T \geq c] = P[\max\{\max\{T_1, T_2\}, T_3 + S_3\} \geq c] \\ = 1 - P[\max\{\max\{T_1, T_2\}, T_3 + S_3\} < c] \\ = 1 - P[\{\max\{T_1, T_2\} < c\} \cdot \{T_3 + S_3 < c\}],$$

where the symbol '$\cdot$' denotes the intersection of two events.

The last probability can be written as follows:

$$P[\{\max\{T_1, T_2\} < c\} \cdot \{T_3 + S_3 < c\}] \\ = P[\{\max\{T_1, T_2\} < c\} \cdot \{T_3 + S_3 < c\} \cdot \{T_1 \leq T_2\}] \\ + P[\{\max\{T_1, T_2\} < c\} \cdot \{T_3 + S_3 < c\} \cdot \{T_1 > T_2\}] \\ = P[\{T_2 < c\} \cdot \{T_3 + T_1 < c\} \cdot \{T_1 \leq T_2\}] \\ + P[\{T_1 < c\} \cdot \{T_3 + T_2 < c\} \cdot \{T_1 > T_2\}].$$

The above two probabilities correspond to figures 2a and 2b, respectively.

Thus,

$$P[\mathcal{S}] = 1 - P[\{T_2 < c\} \cdot \{T_3 + T_1 < c\} \cdot \{T_1 \leq T_2\}] \\ - P[\{T_1 < c\} \cdot \{T_3 + T_2 < c\} \cdot \{T_1 > T_2\}],$$

and therefore with independently distributed lifetimes, we have:

$$P[\mathcal{S}] = 1 - \int_0^c \int_{t_1}^c \int_0^{c-t_1} \left\{ \prod_{i=1}^{3} f_i(t_i) \right\} \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\ - \int_0^c \int_0^{t_1} \int_0^{c-t_2} \left\{ \prod_{i=1}^{3} f_i(t_i) \right\} \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1, \tag{2}$$

where f_i is the PDF of the random variable T_i ($i = 1, 2, 3$).

3.3. *Four independent units*

For $n = 4$, we have:

$$S_1 = S_2 = 0, \quad S_3 = \min\{T_1, T_2\} \quad \text{and} \\ S_4 = \min\{\max\{T_1, T_2\}, T_3 + S_3\}.$$

Thus,

$$T = \max\{\max\{\max\{T_1, T_2\}, T_3 + S_3\}, \\ T_4 + \min\{\max\{T_1, T_2\}, T_3 + S_3\}\}.$$

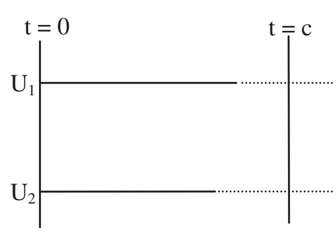

Figure 1. Units U_1 and U_2 starting operation simultaneously at time $t = 0$. The process is not sucessfully controlled for the required time $t = c$.

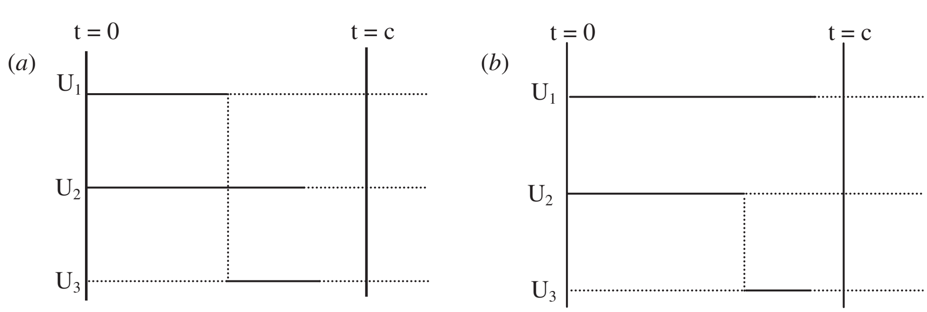

Figure 2. Unit U_3 starting its operation upon the failure time of (a) the unit U_1 and (b) the unit U_2. The process is not successfully controlled for the required time $t = c$.

Since $S_4 = \min\{\max\{T_1, T_2\}, T_3 + S_3\}$, we can write

$$T = \max\{\max\{\max\{T_1, T_2\}, T_3 + S_3\}, T_4 + S_4\},$$

and therefore,

$$\begin{aligned}
P[S] &= P[T \geq c] \\
&= P[\max\{\max\{\max\{T_1, T_2\}, T_3 + S_3\}, T_4 + S_4\} \geq c] \\
&= 1 - P[\max\{\max\{\max\{T_1, T_2\}, T_3 + S_3\}, T_4 + S_4\} < c] \\
&= 1 - P[\{\max\{\max\{T_1, T_2\}, T_3 + S_3\} < c\} \cdot \{T_4 + S_4 < c\}] \\
&= 1 - P[\{\max\{\max\{T_1, T_2\}, T_3 + S_3\} < c\} \cdot \{T_4 + S_4 < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} \leq T_3 + S_3\}] \\
&\quad - P[\{\max\{\max\{T_1, T_2\}, T_3 + S_3\} < c\} \cdot \{T_4 + S_4 < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} > T_3 + S_3\}] \\
&= 1 - P[\{T_3 + S_3 < c\} \cdot \{T_4 + \max\{T_1, T_2\} < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} \leq T_3 + S_3\}] \\
&\quad - P[\{\max\{T_1, T_2\} < c\} \cdot \{T_4 + T_3 + S_3 < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} > T_3 + S_3\}] \\
&= 1 - P[\{T_3 + S_3 < c\} \cdot \{T_4 + \max\{T_1, T_2\} < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} \leq T_3 + S_3\} \cdot \{T_1 \leq T_2\}] \\
&\quad - P[\{T_3 + S_3 < c\} \cdot \{T_4 + \max\{T_1, T_2\} < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} \leq T_3 + S_3\} \cdot \{T_1 > T_2\}] \\
&\quad - P[\{\max\{T_1, T_2\} < c\} \cdot \{T_4 + T_3 + S_3 < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} > T_3 + S_3\} \cdot \{T_1 \leq T_2\}] \\
&\quad - P[\{\max\{T_1, T_2\} < c\} \cdot \{T_4 + T_3 + S_3 < c\} \\
&\qquad \cdot \{\max\{T_1, T_2\} > T_3 + S_3\} \cdot \{T_1 > T_2\}].
\end{aligned}$$

Since now $S_3 = \min\{T_1, T_2\}$, we have $S_3 = T_1$ when $T_1 \leq T_2$, and $S_3 = T_2$ when $T_1 > T_2$.

Thus, we can write alternatively:

$$\begin{aligned}
P[S] = 1 &- P[\{T_3 + T_1 < c\} \cdot \{T_4 + T_2 < c\} \cdot \{T_2 \leq T_3 + T_1\} \\
&\qquad \cdot \{T_1 \leq T_2\}] \\
&- P[\{T_3 + T_2 < c\} \cdot \{T_4 + T_1 < c\} \cdot \{T_1 \leq T_3 + \mathrm{T}_2\} \\
&\qquad \cdot \{T_1 > T_2\}] \\
&- P[\{T_2 < c\} \cdot \{T_4 + T_3 + T_1 < c\} \cdot \{T_2 > T_3 + T_1\} \\
&\qquad \cdot \{T_1 \leq T_2\}] \\
&- P[\{T_1 < c\} \cdot \{T_4 + T_3 + T_2 < c\} \cdot \{T_1 > T_3 + T_2\} \\
&\qquad \cdot \{T_1 > T_2\}].
\end{aligned} \tag{3}$$

Figures 3a–d refer to the four terms above, respectively.

Now, by solving the inequalities within each term above, we get the area of integration of the joint distribution function of the unit lifetimes. Thus, with independently distributed lifetimes, we take:

$$\begin{aligned}
P[S] = 1 &- \int_0^c \int_{t_1}^c \int_{t_2-t_1}^{c-t_1} \int_0^{c-t_2} \left\{ \prod_{i=1}^4 f_i(t_i) \right\} \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\
&- \int_0^c \int_0^{t_1} \int_{t_1-t_2}^{c-t_2} \int_0^{c-t_1} \left\{ \prod_{i=1}^4 f_i(t_i) \right\} \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\
&- \int_0^c \int_{t_1}^c \int_0^{t_2-t_1} \int_0^{c-t_1-t_3} \left\{ \prod_{i=1}^4 f_i(t_i) \right\} \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\
&- \int_0^c \int_0^{t_1} \int_0^{t_1-t_2} \int_0^{c-t_2-t_3} \left\{ \prod_{i=1}^4 f_i(t_i) \right\} \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1,
\end{aligned} \tag{4}$$

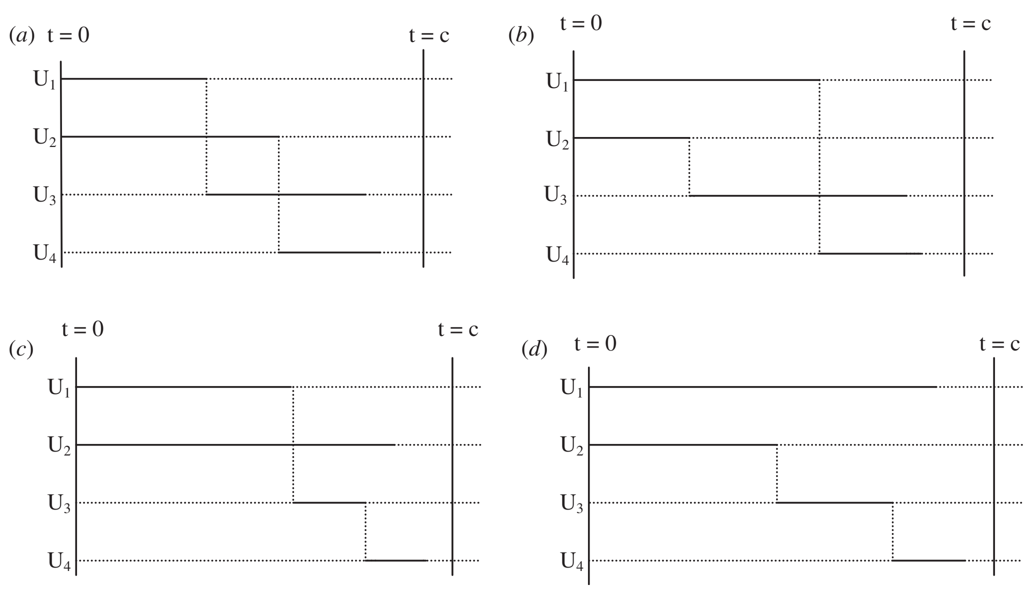

Figure 3. Units U_3 and U_4 starting operation upon the failure time of (a) U_1 and U_2, (b) U_2 and U_1, (c) U_1 and U_3, and (d) U_2 and U_3, respectively. The process is not sucessfully controlled for the required time $t = c$.

where f_i is the PDF of the random variable T_i $(i=1,\ldots,4)$.

3.4. *General case*

We will now derive a recursive relation to evaluate $P[\mathcal{S}]$ for any number, n, of non-independent and non-identically distributed lifetimes.

Here, we have:

$$S_1 = S_2 = 0,$$
$$S_3 = \min\{T_1, T_2\},$$

and in general for $i \geq 4$,

$$S_i = \min\{\max\{T_1, T_2, T_3+S_3, \ldots, T_{i-2}+S_{i-2}\}, T_{i-1}+S_{i-1}\} \quad (i=4,\ldots,n). \tag{5}$$

Let us define now the following random variables:

$$M_2 = T_1,$$
$$M_3 = \max\{T_1, T_2\},$$

and in general for $i \geq 4$.

$$M_i = \max\{T_1, T_2, T_3+S_3, \ldots, T_{i-1}+S_{i-1}\} \quad (i=4,\ldots,n), \tag{6}$$

which, in order to develop recursive relations, will be written in the form

$$M_i = \max\{\max\{T_1, T_2, T_3+S_3, \ldots, T_{i-2}+S_{i-2}\}, T_{i-1}+S_{i-1}\} \quad (i=4,\ldots,n).$$

Then, for the system lifetime, we have

$$T = \max\{T_1, T_2, T_3+S_3, \ldots, T_n+S_n\}.$$

Let us define also

$$V_i = T_i + S_i \quad (i=2,\ldots,n). \tag{7}$$

Thus, we have

$$S_1 = S_2 = 0,$$
$$S_i = \min\{M_{i-1}, V_{i-1}\} \quad (i=3,\ldots,n)$$

and

$$M_i = \max\{M_{i-1}, V_{i-1}\} \quad (i=3,\ldots,n).$$

Consider now the following events:

$$\begin{aligned}
&\text{(i)} && \mathcal{A}_i = \{M_i \leq V_i\} \quad (i=2,\ldots,n-1),\\
&\text{(ii)} && \mathcal{B}_i = \{M_i < c\} \quad (i=2,\ldots,n),\\
&\text{(iii)} && \mathcal{C}_i = \{V_i < c\} \quad (i=2,\ldots,n),\\
&\text{(iv)} && \mathcal{D}_i = \mathcal{B}_i \cap \mathcal{C}_i \quad (i=2,\ldots,n),\\
&\text{(v)} && \mathcal{F}_i = \{T_{i+1} + V_i < c\} \quad (i=2,\ldots,n-1),\\
&\text{(vi)} && \mathcal{G}_i = \{T_{i+1} + M_i < c\} \quad (i=2,\ldots,n-1).
\end{aligned} \tag{8}$$

We may notice now that the complement of the event $\mathcal{D}_n$ is the event, $\mathcal{S}$, that is D_n represents the unsuccessful control of the process by an n-units system for the required period of time $t = c$. Indeed, we have:

$$\begin{aligned}
\mathcal{D}_n &= \mathcal{B}_n \cap \mathcal{C}_n\\
&= \{M_n < c\} \cdot \{V_n < c\} = \{\max\{M_n, V_n\} < c\},
\end{aligned}$$

and introducing (6) for $i = n$, we get

$$\begin{aligned}
\mathcal{D}_n &= \{\max\{T_1, T_2, T_3+S_3, T_4+S_4, \ldots, T_n+S_n\} < c\}\\
&= \{T < c\} = \mathcal{S}',
\end{aligned}$$

where $\mathcal{S}'$ is the complement of the event, $\mathcal{S}$.

Thus,

$$P[\mathcal{S}] = 1 - P[\mathcal{D}_n]. \tag{9}$$

Now, the events $\mathcal{B}_i$ for $i \geq 3$ can be written:

$$\begin{aligned}
\mathcal{B}_i &= \{M_i < c\} = \{\max\{T_1, T_2, \ldots, T_{i-1}+S_{i-1}\} < c\}\\
&= \{\max\{M_{i-1}, V_{i-1}\} < c\} = \{M_{i-1} < c\} \cdot \{V_{i-1} < c\}\\
&= \mathcal{B}_{i-1} \cdot \mathcal{C}_{i-1} = \mathcal{D}_{i-1}.
\end{aligned}$$

Since, by definition, we can write $\mathcal{D}_i = B_i \cdot C_i$ $(i = 2,\ldots,n)$, we have:

$$\mathcal{D}_i = \mathcal{B}_i \cdot \mathcal{C}_i = \mathcal{D}_{i-1} \cdot \mathcal{C}_i \quad (i=3,\ldots,n) \tag{10}$$

with

$$\mathcal{D}_2 = \mathcal{B}_2 \cdot \mathcal{C}_2 = \{T_1 < c\} \cdot \{T_2 < c\}.$$

The event $\mathcal{C}_i$ can be written now as the union of two incompatible events, namely

$$\mathcal{C}_i = \mathcal{C}_i \cdot \mathcal{A}_{i-1} \cup \mathcal{C}_i \cdot \mathcal{A}'_{i-1} \quad (i=3,\ldots,n), \tag{11}$$

and since $S_2 = 0$,

$$\mathcal{C}_2 = \{T_2 + S_2 < c\} = \{T_2 < c\}.$$

In addition,

$$\mathcal{C}_i \cdot \mathcal{A}_{i-1} = \{T_i + \min\{M_{i-1}, V_{i-1}\} < c\} \cdot \{M_{i-1} \le V_{i-1}\}$$
$$= \{T_i + M_{i-1} < c\} \cdot \{M_{i-1} \le V_{i-1}\} = \mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1}$$
$$(i = 3, \dots, n)$$

and

$$\mathcal{C}_i \cdot \mathcal{A}'_{i-1} = \{T_i + \min\{M_{i-1}, V_{i-1}\} < c\} \cdot \{M_{i-1} > V_{i-1}\}$$
$$= \{T_i + V_{i-1} < c\} \cdot \{M_{i-1} > V_{i-1}\} = \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1}$$
$$(i = 3, \dots, n).$$

From the above two results and the relation (11), we have

$$\mathcal{C}_i = \mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1} \cup \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1} \quad (i = 3, \dots, n)$$

with

$$C_2 = \{T_2 < c\}.$$

Thus, from the above result and (10), the event $\mathcal{D}_i$ with i ≥ 3 can be written

$$\mathcal{D}_i = (\mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1} \cup \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1}) \cdot \mathcal{D}_{i-1} \quad (i = 3, \dots, n), \tag{12}$$

where the events $\mathcal{G}_i$, $\mathcal{F}_i$, and $\mathcal{A}_i$ are given by (8).

Using the above recursive relation for $i = n$, we have:

$$\mathcal{D}_n = \left\{ \bigcap_{i=3}^{n} (\mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1} \cup \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1}) \right\} \cdot \mathcal{D}_2. \tag{13}$$

We now set $\mathcal{W}_i$ and $\mathcal{Q}_i$ the intersections $\mathcal{G}_i \cdot \mathcal{A}_i$ and $\mathcal{F}_i \cdot \mathcal{A}'_i$, respectively, for $i = 2, \dots, n-1$. Since the events $\mathcal{W}_i$ and $\mathcal{Q}_i$ are incompatible, the probability of the event $\mathcal{D}_n$ can be written:

$$P[\mathcal{D}_n] = P\left[\left\{ \bigcap_{i=2}^{n-1} (\mathcal{G}_i \cdot \mathcal{A}_i \cup \mathcal{F}_i \cdot \mathcal{A}'_i) \right\} \cdot \mathcal{D}_2 \right]$$
$$= P\left[\left\{ \bigcap_{i=2}^{n-1} (\mathcal{W}_i \cup \mathcal{Q}_i) \right\} \cdot \mathcal{D}_2 \right]$$
$$= P\left[\bigcup_{j=1}^{2^{n-2}} \bigcap_{i=1}^{n-1} \mathcal{Y}_i^{(j)} \right]$$

or

$$P[\mathcal{D}_n] = \sum_{j=1}^{2^{n-2}} P\left[\bigcap_{i=1}^{n-1} \mathcal{Y}_i^{(j)} \right], \tag{14}$$

where $\mathcal{Y}_1^{(j)} = \mathcal{D}_2$, and for $i = 2, \dots, n-1$, $\mathcal{Y}_i^{(j)}$ is either $\mathcal{W}_i = \mathcal{G}_i \cdot \mathcal{A}_i$ or $\mathcal{Q}_i = \mathcal{F}_i \cdot \mathcal{A}'_i$ as j runs from 1 to 2^{n-2} covering all possible choices.

For the probability of the event $\mathcal{D}_n$, we have:

$$P[\mathcal{D}_n] = \sum_{j=1}^{2^{n-2}} P\left[\bigcap_{i=1}^{n-1} \mathcal{Y}_i^{(j)} \right] = \sum_{j=1}^{2^{n-2}} \left[\int_{\mathcal{K}_j} \mathrm{d}F(\mathbf{t}) \right], \tag{15}$$

where $\mathcal{K}_j$ is an appropriate region in $\Re^n$ specified below.

From the definition of $\mathcal{Y}_i^{(j)}$ it can be proved that the region $\mathcal{K}_j$ is given by:

$$\mathcal{K}_j = \mathcal{K}_{1j} \times \cdots \times \mathcal{K}_{nj} \quad (j = 1, \dots, 2^{n-2}), \tag{16}$$

where, for all j, $\mathcal{K}_{ij}$ $(i = 1, \dots, n)$ are intervals for the variable t_i, respectively, which are defined by the variables $t_1, \dots, t_{i-1}$ as follows:

$$\text{(i)} \quad \mathcal{K}_{1j} = \{t_1 < c\},$$
$$\text{(ii)} \quad \mathcal{K}_{ij} = [\{M_i - S_i \le t_i < c - S_i\} \cdot I(\mathcal{Y}_i^{(j)}, \mathcal{W}_i)] \cup [\{0 \le t_i < M_i - S_i\} \cdot I(\mathcal{Y}_i^{(j)}, \mathcal{Q}_i)] \quad (i = 2, \dots, n-1),$$
$$\text{(iii)} \quad \mathcal{K}_{nj} = [\{0 \le t_n < c - M_{n-1}\} \cdot I(\mathcal{Y}_{n-1}^{(j)}, \mathcal{W}_{n-1})] \cup [\{0 \le t_n < c - V_{n-1}\} \cdot I(\mathcal{Y}_{n-1}^{(j)}, \mathcal{Q}_{n-1})], \tag{17}$$

where I stands for the indicator function for sets, i.e. $I(\mathcal{Y}, \mathcal{Q}) = \Omega$ when $\mathcal{Y} = \mathcal{Q}$ and $I(\mathcal{Y}, \mathcal{Q}) = \emptyset$ when $\mathcal{Y} \neq \mathcal{Q}$.

Thus, from (15) with absolutely continuous lifetime distributions, the probability of the event $\mathcal{D}_n$, i.e. the probability of the unsuccessful control with n units, is given by:

$$P[\mathcal{D}_n] = \sum_{j=1}^{2^{n-2}} \int_{\mathcal{K}_j} \mathrm{d}F(\mathbf{t}) = \sum_{j=1}^{2^{n-2}} \int_{\mathcal{K}_j} f(\mathbf{t})\,\mathrm{d}\mathbf{t}, \tag{18}$$

and after introducing (16) into (18) we take

$$P[\mathcal{D}_n] = \sum_{j=1}^{2^{n-2}} \int_{\mathcal{K}_{1j}} \cdots \int_{\mathcal{K}_{nj}} f(\mathbf{t})\,\mathrm{d}t_n \dots \mathrm{d}t_1 \tag{19}$$

with $\mathcal{K}_{ij}$ $(i = 1, \dots, n;\ j = 1, \dots, 2^{n-2})$ given by (17).

4. Applications

Here, we give the final expressions for the system reliability applying the results of the previous section for n = 3, 4, and 5.

4.1. *Three-non-identical units parallel system*

For $n = 3$, we have from (9)

$$P[\mathcal{S}] = P[T \geq c] = 1 - P[\mathcal{D}_3], \text{ with } P[\mathcal{D}_3] \text{ given by (14)},$$

namely

$$P[\mathcal{D}_3] = \sum_{j=1}^{2} P\left[\bigcap_{i=1}^{2} \mathcal{Y}_i^{(j)}\right],$$

where $\mathcal{Y}_1^{(j)} = \mathcal{D}_2$, and $\mathcal{Y}_2^{(j)}$ is $\mathcal{W}_2 = \mathcal{G}_2 \cdot \mathcal{A}_2$ for $j = 1$ and $\mathcal{Q}_2 = \mathcal{F}_2 \cdot \mathcal{A}_2'$ for $j = 2$.

Using (19), the probability of the event $\mathcal{D}_3$ is given by:

$$\begin{aligned} P[\mathcal{D}_3] &= P[\mathcal{D}_2\mathcal{W}_2] + P[\mathcal{D}_2\mathcal{Q}_2] \\ &= \sum_{j=1}^{2} \int_{\mathcal{K}_{1j}} \int_{\mathcal{K}_{2j}} \int_{\mathcal{K}_{3j}} f(\mathbf{t})\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1. \end{aligned}$$

Now, the sets $\mathcal{K}_{ij}$ $(i = 1, 2, 3; \quad j = 1, 2)$ from (17) here are:

$$\begin{aligned} \mathcal{K}_{1j} &= \{t_1 < c\}, \\ \mathcal{K}_{2j} &= [\{M_2 - S_2 \leq t_2 < c - S_2\} \cdot I(\mathcal{Y}_2^{(j)}, \mathcal{W}_2)] \\ &\quad \cup [\{0 \leq t_2 < M_2 - S_2\} \cdot I(\mathcal{Y}_2^{(j)}, \mathcal{Q}_2)], \\ \mathcal{K}_{3j} &= [\{0 \leq t_3 < c - M_2\} \cdot I(\mathcal{Y}_2^{(j)}, \mathcal{W}_2)] \\ &\quad \cup [\{0 \leq t_3 < c - V_2\} \cdot I(\mathcal{Y}_2^{(j)}, \mathcal{Q}_2)] \end{aligned}$$

with

$$S_1 = S_2 = 0, \quad M_2 = T_1, \quad V_2 = T_2.$$

Therefore, we have

$$\begin{aligned} P[\mathcal{S}] &= 1 - P[\mathcal{D}_3] \\ &= 1 - \left\{ \int_0^c \int_{t_1}^c \int_0^{c-t_1} f(\mathbf{t})\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \right. \\ &\quad \left. + \int_0^c \int_0^{t_1} \int_0^{c-t_2} f(\mathbf{t})\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \right\}, \end{aligned} \tag{20}$$

which coincides with (2) for independent lifetimes.

4.2. *Four-non-identical units parallel system*

For $n = 4$, we have

$$P[\mathcal{S}] = P[T \geq c] = 1 - P[\mathcal{D}_4]$$

with

$$P[\mathcal{D}_4] = \sum_{j=1}^{4} P\left[\bigcap_{i=1}^{3} \mathcal{Y}_i^{(j)}\right],$$

where $\mathcal{Y}_1^{(j)} = \mathcal{D}_2$ and for $i = 2, 3$ the event $\mathcal{Y}_i^{(j)}$ is either $\mathcal{W}_i$ or $\mathcal{Q}_i$ as j runs from 1 to 4.

Thus, using (19), the probability of the event $\mathcal{D}_4$ is given by:

$$\begin{aligned} P[\mathcal{D}_4] &= P[\mathcal{D}_2\mathcal{W}_2\mathcal{W}_3] + P[\mathcal{D}_2\mathcal{Q}_2\mathcal{W}_3] + P[\mathcal{D}_2\mathcal{W}_2\mathcal{Q}_3] \\ &\quad + P[\mathcal{D}_2\mathcal{Q}_2\mathcal{Q}_3] \\ &= \sum_{j=1}^{4} \int_{\mathcal{K}_{1j}} \int_{\mathcal{K}_{2j}} \int_{\mathcal{K}_{3j}} \int_{\mathcal{K}_{4j}} f(\mathbf{t})\, \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1. \end{aligned}$$

Now, from the relations (17), the sets $\mathcal{K}_{ij}$ $(i = 1, \ldots, 4;$ $j = 1, \ldots, 4)$ are:

$$\begin{aligned} \mathcal{K}_{1j} &= \{t_1 < c\}, \\ \mathcal{K}_{ij} &= [\{M_i - S_i \leq t_i < c - S_i\} \cdot I(\mathcal{Y}_i^{(j)}, \mathcal{W}_i)] \\ &\quad \cup [\{0 \leq t_i < M_i - S_i\} \cdot I(\mathcal{Y}_i^{(j)}, \mathcal{Q}_i)] \quad (i = 2, 3), \\ \mathcal{K}_{4j} &= [\{0 \leq t_4 < c - M_3\} \cdot I(\mathcal{Y}_3^{(j)}, \mathcal{W}_3)] \\ &\quad \cup [\{0 \leq t_4 < c - V_3\} \cdot I(\mathcal{Y}_3^{(j)}, \mathcal{Q}_3)] \end{aligned}$$

with

$$S_1 = S_2 = 0, \quad M_2 = T_1, \quad V_2 = T_2,$$

and

$$S_3 = \min\{T_1, T_2\}, \; M_3 = \max\{T_1, T_2\}, \; V_3 = T_3 + S_3.$$

Therefore, we have

$$\begin{aligned} P[\mathcal{S}] &= 1 - P[\mathcal{D}_4] \\ &= 1 - \left\{ \int_0^c \int_{t_1}^c \int_{t_2-t_1}^{c-t_1} \int_0^{c-t_2} f(\mathbf{t})\, \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \right. \\ &\quad + \int_0^c \int_0^{t_1} \int_{t_1-t_2}^{c-t_2} \int_0^{c-t_1} f(\mathbf{t})\, \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\ &\quad + \int_0^c \int_{t_1}^c \int_0^{t_2-t_1} \int_0^{c-t_1-t_3} f(\mathbf{t})\, \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\ &\quad \left. + \int_0^c \int_0^{t_1} \int_0^{t_1-t_2} \int_0^{c-t_2-t_3} f(\mathbf{t})\, \mathrm{d}t_4\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \right\}, \end{aligned} \tag{21}$$

which coincides with (4) for independent lifetimes.

4.3. *Five-non-identical units parallel system*

For higher values of n, the evaluation of the probability of the successful control of the process becomes quite complicated. The results (9), (14), (17), and (19) are quite helpful in deriving an explicit expression for the above probability. Applying these results for $n = 5$, we directly get the following expression for $P[\mathcal{S}]$.

$$\begin{aligned}
P[\mathcal{S}] &= 1 - P[\mathcal{D}_5] \\
&= 1 - \left\{ \int_0^c \int_{t_1}^c \int_{t_2-t_1}^{c-t_1} \int_{t_3-t_2+t_1}^{c-t_2} \int_0^{c-t_1-t_3} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \right. \\
&\quad + \int_0^c \int_0^{t_1} \int_{t_1-t_2}^{c-t_2} \int_{t_3+t_2-t_1}^{c-t_1} \int_0^{c-t_2-t_3} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\quad + \int_0^c \int_{t_1}^c \int_0^{t_2-t_1} \int_{t_2-t_3-t_1}^{c-t_3-t_1} \int_0^{c-t_2} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\quad + \int_0^c \int_0^{t_1} \int_0^{t_1-t_2} \int_{t_1-t_2-t_3}^{c-t_2-t_3} \int_0^{c-t_1} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\quad + \int_0^c \int_{t_1}^c \int_{t_2-t_1}^{c-t_1} \int_0^{t_3-t_2+t_1} \int_0^{c-t_2-t_4} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\quad + \int_0^c \int_0^{t_1} \int_{t_1-t_2}^{c-t_2} \int_0^{t_3+t_2-t_1} \int_0^{c-t_1-t_4} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\quad + \int_0^c \int_{t_1}^c \int_0^{t_2-t_1} \int_0^{t_2-t_3-t_1} \int_0^{c-t_1-t_3-t_4} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\quad \left. + \int_0^c \int_0^{t_1} \int_0^{t_1-t_2} \int_0^{t_1-t_2-t_3} \int_0^{c-t_2-t_3-t_4} f(\mathbf{t})\,\mathrm{d}t_5\,\mathrm{d}t_4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \right\}.
\end{aligned} \tag{22}$$

The above expression can provide the final value for $P[\mathcal{S}]$ in a straightforward way when the unit lifetimes are, for example, independent and exponentially distributed. For less mathematically tractable distributions, or larger values of n, mathematical packages like *Mathematica* or *Matlab* can be applied.

The result (14) together with the recursive relation (12) provides an explicit expression for the probability of $\mathcal{D}_n$ as a sum of probabilities involving simple expressions for the unit lifetimes in terms of inequalities provided by the expressions (i)–(vi) in (8). For example, the application of the above results for $n = 4$ directly provides the relation (3).

Remark 1: Based on those inequalities, the reliability of the system and the expected number of units used can be easily evaluated by applying Markov Chains Monte Carlo (MCMC) techniques. The expected number $E[N]$ of units N used during the required period of time, c, is evaluated through the expression: $E[N] = n\{1 - P[\mathcal{S}]\} + E[N|\mathcal{S}]\ P[\mathcal{S}]$. We have developed a simulation program that runs with *Mathematica* and provides final values for the system reliability and the expected number of units used for any value of n with independent lifetimes and general lifetime distributions.

5. Analytical results with special distributions

In this section, we present analytical results for the reliability of an n-unit system with $n = 3$, 4, and 5 with independent unit lifetimes identically and exponentially distributed. For the non-identical case, we present a three-independent-units system with unit lifetimes exponentially distributed. We also present a three-identical-independent unit system with unit lifetimes following a certain Weibull distribution.

5.1. *Reliability of a three-identical-units system with exponential lifetimes*

If the lifetimes of the three units are independent exponentially distributed with the same parameter b, then from (2), or equivalently from (20), we get the following result:

$$P[\mathcal{S}] = P[T \geq c] = -3e^{-2bc} - 2bce^{-2bc} + 4e^{-bc}. \tag{23}$$

The system reliability (23) for various values of c and b ($c \in [0, 100]$ and $b \in [0.02, 0.04]$), is shown in figure 4.

According to Kokolakis *et al.* (1996) where all three units are allowed to work in parallel, and the unit starting times S_i are prescheduled and equal to $s_i \quad (i = 2, 3)$, respectively, with $s_1 = 0$, the corresponding expression for the system reliability is as follows:

$$\begin{aligned}
P[\mathcal{S}] = P[T \geq c] = 4e^{-bc} - e^{-b(2c-s_2)} - 2e^{-b(2c-s_3)} \\
- e^{-b(c+s_3-s_2)} + e^{-b(3c-s_2-s_3)}.
\end{aligned} \tag{24}$$

After optimizing the above expression with respect to s_2 and s_3, we get $s_2 = 0$ and $s_3 = c - (1/2b)\ln(2e^{2b} - 1)$ $(\leq c)$.

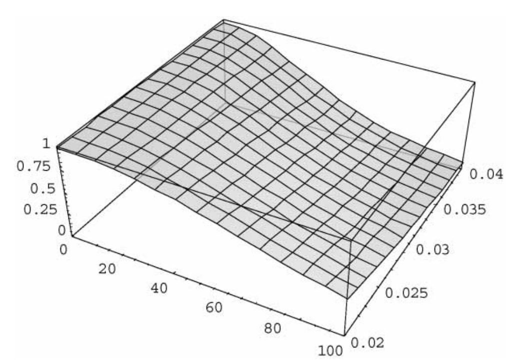

Figure 4. System reliability for required period $c \in [0, 100]$ with parameter $b \in [0.02, 0.04]$.

Substituting these values in the relation (24), we have:

$$
\begin{aligned}
P[S] &= P[T \geq c] \\
&= 4e^{-bc} - e^{-2bc} - 2\exp\left\{-b\left[c + \frac{\ln(-1+2e^{bc})}{2b}\right]\right\} \\
&\quad - \exp\left\{-b\left[2c - \frac{\ln(-1+2e^{bc})}{2b}\right]\right\} \\
&\quad + \exp\left\{-b\left[2c + \frac{\ln(-1+2e^{bc})}{2b}\right]\right\}.
\end{aligned} \tag{25}
$$

Figure 5 presents reliability diagrams for $c = 10, 20, 30, 40, 50$, and 60, and for $b \in [0.02, 0.04]$ for the above two policies. It is obvious that the second policy is inferior, as expected, to our policy, and this is due to the fact that the second policy refers to an open loop problem where up to n units are allowed to work simultaneously.

5.2. *Reliability of a four-identical-units system with exponential lifetimes*

If the lifetimes of the four units are independently exponentially distributed with the same parameter b,

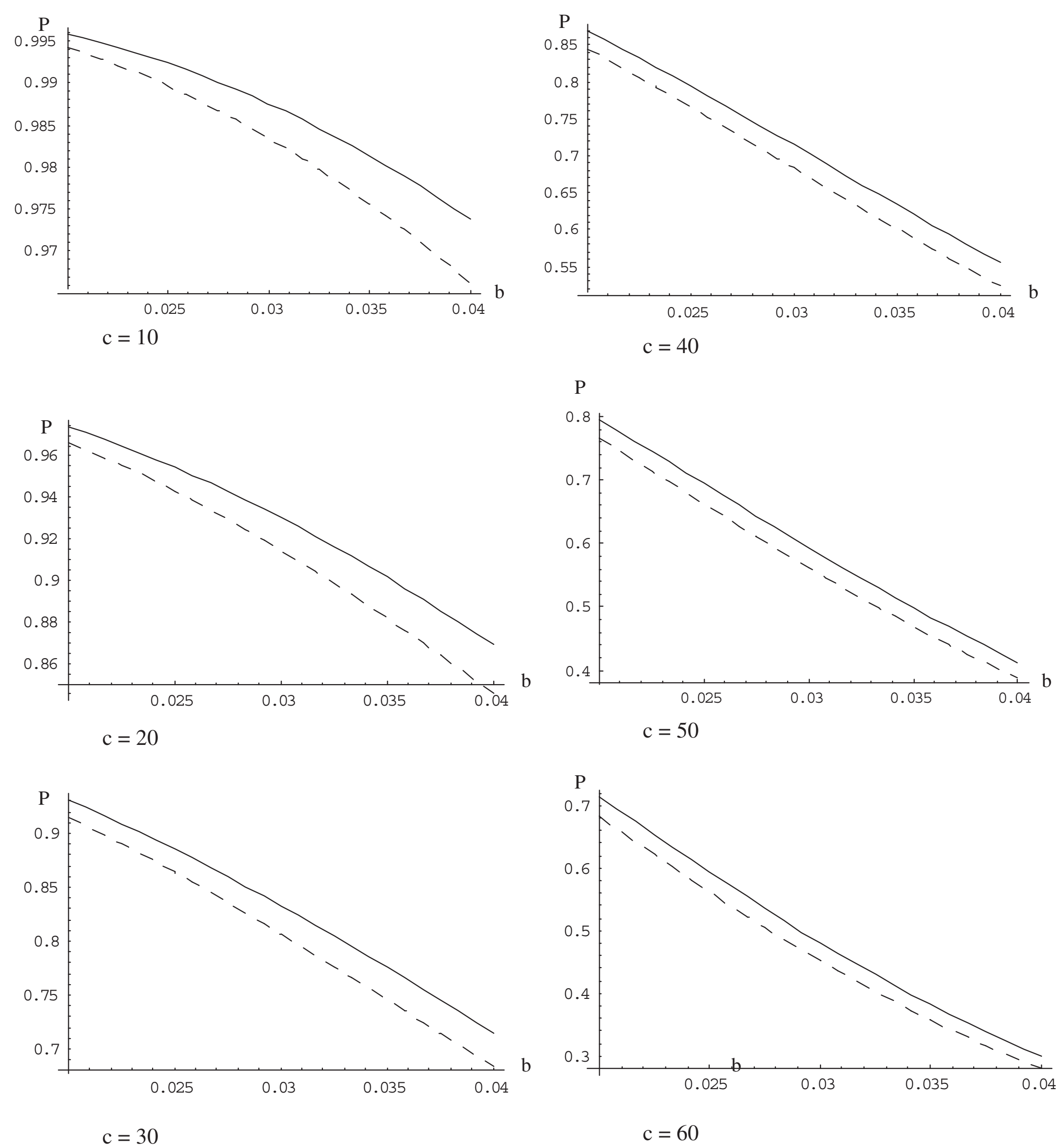

Figure 5. System reliability diagrams for required period $c = 10, 20, 30, 40, 50$, and 60, and $b \in [0.02, 0.04]$. The dotted lines correspond to the second policy (formula 25).

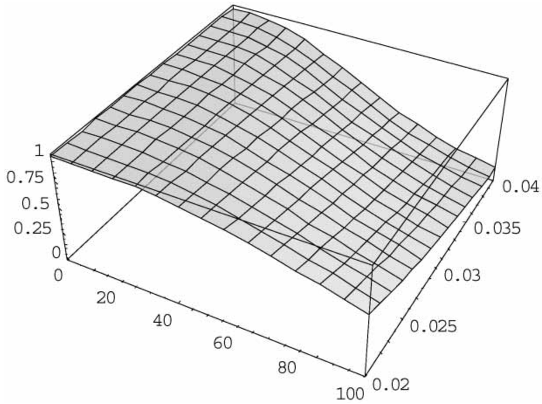

Figure 6. System reliability for required period $c \in [0, 100]$ with parameter $b \in [0.02, 0.04]$.

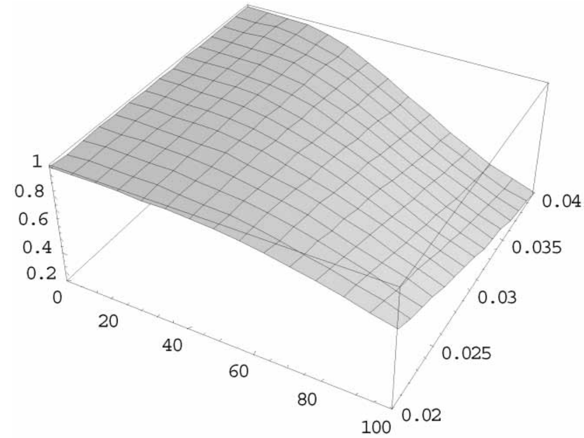

Figure 7. System reliability for required period $c \in [0, 100]$ with parameter $b \in [0.02, 0.04]$.

then from (4), or equivalently from (21), we have the following result:

$$P[\mathcal{S}] = P[T \geq c] = -7e^{-2bc} - 6bce^{-2bc} - 2b^2c^2e^{-2bc} + 8e^{-bc}. \tag{26}$$

The reliability (26) for various values of c and b ($c \in [0, 100]$ and $b \in [0.02, 0.04]$) is shown in the following figure 6.

5.3. *Reliability of a five-identical-units system with exponential lifetimes*

If the lifetimes of the five units are independent exponentially distributed with the same parameter b, then from (22), we have the following result:

$$P[\mathcal{S}] = P[T \geq c] = -15e^{-2bc} - 14bce^{-2bc} - 6b^2c^2e^{-2bc} - \frac{4}{3}b^3c^3e^{-2bc} + 16e^{-bc}. \tag{27}$$

The reliability (27) for various values of c and b ($c \in [0, 100]$ and $b \in [0.02, 0.04]$) is shown in figure 7.

5.4. *Reliability of a three-non-identical units system with exponential lifetimes*

If the lifetimes of the three units are independently exponentially distributed with different parameters b_i $(i = 1, 2, 3)$, the system reliability $P[\mathcal{S}]$ is derived applying the result (2) or equivalently (20). The final expression for $P[\mathcal{S}]$ is quite large and thus is presented in the Appendix.

Remark 2: It is interesting to notice that the expression for $P[\mathcal{S}]$ in the Appendix is symmetrical with respect to b_1 and b_2. This is also true with any lifetime distributions and is due to the fact that the first two units start simultaneously.

In the case of different lifetime distributions, it might be of interest to optimize the system with respect to the order in which the units are introduced. In the case of three non-identical units with exponential lifetimes, it can be proved easily from the expression provided in the Appendix that the probability of successful control is maximized by using the best, i.e. the one with the larger mean lifetime, last. For example, with $c = 10$ and (b_1, b_2, b_3), the permutations of the numbers 0.08, 0.05, and 0.025, we obtain $P[\mathcal{S}] = 0.9647$ when $b_3 = 0.025$, $P[\mathcal{S}] = 0.9642$ when $b_3 = 0.05$ and $P[\mathcal{S}] = 0.9636$ when $b_3 = 0.08$, while the interchange of b_1 with b_2 does not matter.

In addition, analytical results from (21), or equivalently from (4), in the case of four non-identical units with exponential lifetimes, have been derived. Numerical examples show again that the better unit has to be used later.

5.5. *Reliability of a three-identical-units system with Weibull lifetimes*

If the lifetimes of the three units are independent and follow a Weibull distribution with the same parameters $b = 2$ and $\lambda = 1$, i.e. when $f_i(t) = 2te^{-t^2}$ $(i = 1, 2, 3)$, then from (2), or equivalently from (20), we have the following result:

$$\begin{aligned} P[\mathcal{S}] &= P[T \geq c] \\ &= -\frac{5}{3}e^{-2c^2} + \frac{8}{3}e^{-c^2} - ce^{-(3/2)c^2}\sqrt{2\pi}\, Erf\left[\frac{c}{\sqrt{2}}\right] \\ &\quad + \frac{2}{3}ce^{-(2/3)c^2}\sqrt{\frac{\pi}{3}}\, Erf\left[\frac{c}{\sqrt{3}}\right] + \frac{2}{3}ce^{-(2/3)c^2}\sqrt{\frac{\pi}{3}}\, Erf\left[\frac{2c}{\sqrt{3}}\right], \end{aligned} \tag{28}$$

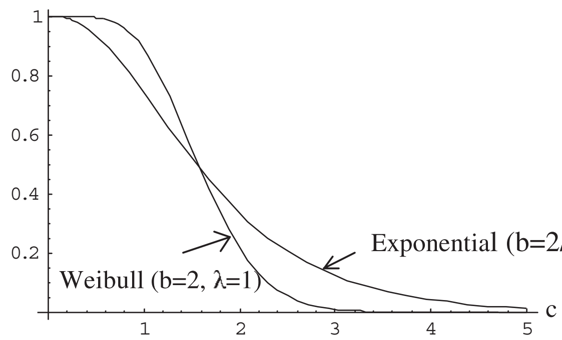

Figure 8. System reliability functions for required-period $c \in [0, 5]$ of a three-units system with Weibull and exponential unit lifetimes distributions of equal means.

where

$$Erf[z] = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \mathrm{d}t.$$

In this case, the units' mean lifetimes are $E[T_i] = \sqrt{\pi}/2$ ($i = 1, 2, 3$). It is interesting to notice that if the lifetimes of the three units above had an exponential distribution with the same mean, i.e. with $b^{-1} = \sqrt{\pi}/2$, the system reliability is lower for small values of c and higher for large values of c, as figure 8 shows.

6. Remarks and conclusions

This paper presents the determination of the reliability of a two-unit redundant system supported by $(n-2)$ standbys by using recursive probabilistic analysis. Units are non-repairable with general and non-identical lifetimes. Our basic results refer to non-independent unit lifetimes, while independence is considered as a special case. Applying special distributions, the system reliability is evaluated in a straightforward way in many cases. The evaluation of the system reliability can be complicated for mathematically intractable distributions or for large values of n. In future, this work can be extended, applying simulation programs or techniques like Markov Chains Monte Carlo (MCMC), which can be proved very helpful. Also, it can be useful in case of non-identical distributions to investigate the problem of optimization with respect to the ordering of the units. Another interesting extension of this work is to apply an analogous probabilistic analysis to a three-unit redundant system supported by $(n-3)$ standbys.

Acknowledgements

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Appendix

Applying the result (2), or equivalently (20), for independent lifetimes of three units which follow exponential distributions with parameters b_i ($i = 1, 2, 3$), we get the following result for the system reliability $P[S]$:

$$\begin{aligned}
P[S] = P[T \geq c] = 1 & - \frac{b_1^3}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& - \frac{b_2^3}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& - \frac{b_1^2 b_2}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& - \frac{b_2^2 b_1}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& + \frac{2b_1^2 b_3}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& + \frac{2b_2^2 b_3}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& + \frac{b_1 b_2 b_3}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& + \frac{b_1 b_2 b_3}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& - \frac{b_1 b_3^2}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& - \frac{b_2 b_3^2}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& + \frac{b_2^2 b_1 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& + \frac{b_1^2 b_2 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& + \frac{2b_1^2 b_2 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& + \frac{2b_1 b_2^2 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& + \frac{b_1^3 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& + \frac{b_2^3 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& - \frac{b_2^2 b_3 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} \\
& - \frac{b_1^2 b_3 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
& - \frac{3b_1 b_2 b_3 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3b_1b_2b_3e^{-b_1c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
-\frac{2b_1^2b_3e^{-b_2c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
-\frac{2b_2^2b_3e^{-b_1c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&+\frac{b_2b_3^2e^{-b_2c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
+\frac{b_1b_3^2e^{-b_1c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
+\frac{b_1b_3^2e^{-b_2c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}\\
&+\frac{b_2b_3^2e^{-b_1c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
+\frac{b_2^2b_3e^{-(b_1+b_2)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
+\frac{b_1^2b_3e^{-(b_1+b_2)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&+\frac{2b_1b_2b_3e^{-(b_1+b_2)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
+\frac{2b_1b_2b_3e^{-(b_1+b_2)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
-\frac{b_2b_3^2e^{-(b_1+b_2)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}\\
&-\frac{b_1b_3^2e^{-(b_1+b_2)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
-\frac{b_1^3e^{-b_3c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
-\frac{b_2^3e^{-b_3c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&+\frac{b_1^2b_2e^{-b_3c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
+\frac{b_1b_2^2e^{-b_3c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
-\frac{b_1^2b_3e^{-b_3c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}\\
&-\frac{b_2^2b_3e^{-b_3c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
-\frac{b_1b_2b_2e^{-b_3c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
-\frac{b_1b_2b_2e^{-b_3c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&-\frac{b_1^3e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
-\frac{b_2^3e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
-\frac{2b_1^2b_2e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}\\
&-\frac{2b_1b_2^2e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
-\frac{b_1b_2^2e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
-\frac{b_1^2b_2e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&+\frac{b_1^2b_3e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}
+\frac{b_2^2b_3e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
+\frac{b_1b_2b_3e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}\\
&+\frac{b_1b_2b_3e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}.
\end{aligned}
$$

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