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A two-unit general parallel system with $(n-2)$ cold standbys—Analytic and simulation approach

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Abstract

A parallel $(2, n-2)$-system is investigated here where two units start their operation simultaneously and any one of them is replaced instantaneously upon its failure by one of the $(n-2)$ cold standbys. We assume availability of n non-identical, non-repairable units for replacement or support. The system reliability is evaluated by recursive relations with unit-lifetimes $T_i\,(i = 1, \ldots, n)$ that have a general joint distribution function $F(\mathbf{t})$. On the basis of the derived expression, simulation techniques have been developed for the evaluation of the system reliability and the mean time to failure, useful when dealing with large systems or correlated unit-lifetimes and less mathematically manageable distributions. Simulation results are presented for various lifetime distributions and comparisons are made with derived analytic results for some special distributions and moderate values of n.

Keywords: Reliability; Simulation; Redundancy; Cold standbys; Correlated lifetimes; General distributions; Rejection method

1. Introduction

Repairable two-unit redundant systems have attracted the attention of several researchers working in the field of reliability theory. Many workers, including Murari and Goel [10], Gupta and Goel [3], Goel et al. [1] and Gupta and Chaudhary [5], have investigated the two-unit standby system models assuming that, on the failure of one unit, it is replaced by the standby unit instantaneously. Gupta and Kishan [6] consider situations of a two-unit system where the standby unit does not operate instantaneously but a fixed preparation time is required to put standby and repaired units into operation. Several workers including Gopalan et al. [2], Gupta and Goel [4], Murari and Maruthachalam [9], have analysed two-unit system models under a

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Nomenclature

U_i	the ith unit introduced into the system
T_i	lifetime of ith unit
$f_i(\cdot)$, $F_i(\cdot)$	PDF and CDF of the ith unit-lifetime
$f(\mathbf{t})$, $F(\mathbf{t})$	joint PDF and CDF of unit-lifetimes
$R(\cdot)$	reliability function
S_i	starting time of operation of the ith unit
n	number of available and non-repairable units
T	system lifetime
MTSF	mean time to system failure
c	fixed required period of system operation
$\mathscr{S}$	the event $\{T \geqslant c\}$

variety of assumptions using the regenerative point technique. In all these system models it is assumed that the lifetimes are uncorrelated random variables. Gupta et al. [7] have investigated a two non-identical parallel system with correlated lifetimes and independently distributed repair times. In these models the repair time may be fixed or random and with or without administrative delay. In all these parallel system models it is assumed that the system failure occurs only when both units stop functioning but it is not necessary that both units have to work simultaneously. The structure of these system models allows the operation of only one unit during the repair time of the other.

In Kokolakis et al. [8], a multi-unit non-repairable redundant parallel system is considered. The units start at prescheduled times and up to n units may be working simultaneously. The system reliability is derived for independent lifetimes with general distributions and its optimization with respect to the units' prescheduled starting times is investigated. Utkin [13] studied the reliability of parallel systems and provided bounds for the reliability under partial information about lifetime distributions and independence.

Papageorgiou and Kokolakis [11], have investigated a two-unit parallel system supported by standbys. In that system, there is available a fixed number of n, non-repairable and non-identical units. Two units operate simultaneously until, at least, the entrance of the last standby unit. The two-unit parallel system is up, as long as there is at least one working unit. The unit lifetimes are considered random, not necessarily independently distributed, with a general joint distribution F. The main result there is the analytic evaluation of the system reliability, unlike most earlier results which provide bounds under partial information about the joint pdf, or refer to specific independent unit lifetime distributions. The system reliability is provided in an exact closed form in cases of some special lifetimes distributions and moderate values of n. Using an analogous probabilistic analysis in this paper, we extend the above main result by deriving a reliability formula which is efficient and easy to use for simulation techniques. These techniques are necessary for large systems, or for correlated unit-lifetimes, or for mathematically intractable distributions, cases where the result in [11], does not provide a closed form for the system reliability. On the basis of our main result an algorithm is developed for the evaluation of the system reliability and the mean time to system failure (MTSF) and simulated results are presented. A comparison between the derived analytic and simulated results for the system reliability and MTSF underlines the usefulness of this investigation.

2. System description

The problem of the successful control of a process by a two-unit parallel system supported by cold standbys is considered in this paper.

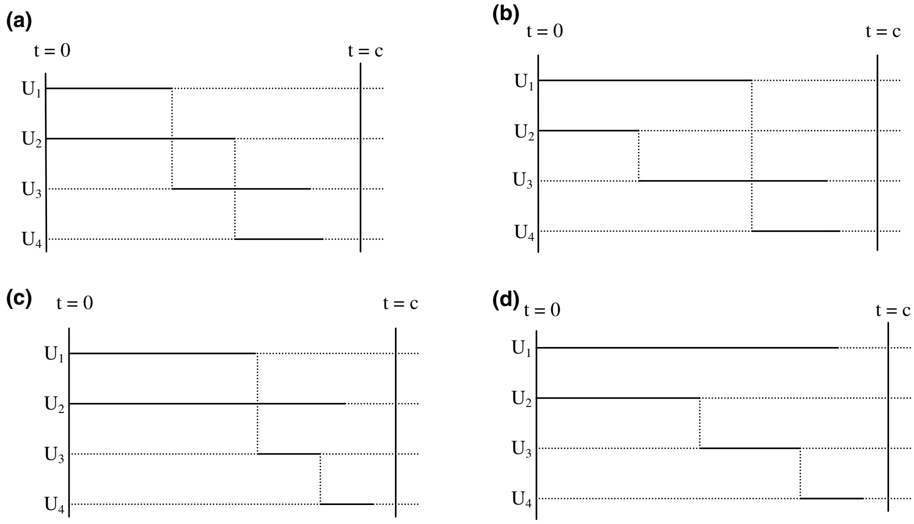

Fig. 1. Units U_3 and U_4 starting operation upon the failure time of the (a) U_1 and U_2, (b) U_2 and U_1, (c) U_1 and U_3, and (d) U_2 and U_3, respectively. The process is not successfully controlled for the required time $t = c$.

Here, we suppose that there is available a fixed number of n non-repairable and non-identical units. The unit-lifetimes T_i $(i = 1, \ldots, n)$ are random, not necessarily independently distributed, with a general joint distribution F. The process has a fixed duration c. Thus, the control of the process is considered successful when the system is up, i.e. at least one of its two units is in operation state, during the required time interval c. The process is initially controlled by two units and the remaining $(n - 2)$ units are cold standbys. The two initial units start their operation simultaneously, i.e. at times $S_1 = S_2 = 0$. The first failed unit is replaced upon its failure instantaneously by a new one, i.e. the third unit starts its operation at time $S_3 = \min\{T_1, T_2\}$. Similarly the forth unit starts its operation upon the failure time of the first failed unit among the two working ones, i.e. at time $S_4 = \min\{\max\{T_1, T_2\}, S_3 + T_3\}$ (Fig. 1).

In general, the ith unit starts its operation at time $S_i = \min\{\max\{T_1, T_2, S_3 + T_3, \ldots, S_{i-2} + T_{i-2}\}, S_{i-1} + T_{i-1}\}$, and it works in parallel with that already introduced and not failed before S_i. Thus, the times S_i $(i = 3, 4, \ldots, n)$ are random, and the process is simultaneously controlled by two working units until, at least, the entrance of the last available unit. The system fails when all units fail and the system reliability depends entirely on the joint distribution of unit-lifetimes.

There are many realistic situations where the above policy describes the control of a process by a parallel system. For example, a satellite communication system with two satellites in orbit and $(n - 2)$ standbys. If at least one of two satellites is functioning in orbit then the system is considered a success, while if no-one satellite functions in orbit, the system is considered a failure. It is assumed that the time required to place a satellite in orbit after launch is negligible. Another example, is a power station which supports a small region or an island. The power station is always functioning with two generators till the stock's spending.

3. Model analysis

Let T, be the system lifetime and T_i $(i = 1, \ldots, n)$, be the lifetimes of the units U_i, with joint distribution, F. Let also $\mathcal{S}$, be the event of the successful control of the process during the required period of time c, i.e. $\mathcal{S} = \{T \geqslant c\}$.

From the description of the policy above, it follows that the system is non-stop functioning with two units until, at least, the entrance of the unit U_n at time $S_n < c$. We are interested in evaluating the system reliability $P[\mathscr{S}] = P[T \geqslant c]$ and the mean time to system failure (MTSF).

3.1. General case

We will derive a recursive relation to evaluate $P[\mathscr{S}]$ for any number, n, of non-independent and non-identically distributed lifetimes.

Here, we have:

$$S_1 = S_2 = 0,$$
$$S_3 = \min\{T_1, T_2\},$$

and in general for $i \geqslant 4$,

$$S_i = \min\{\max\{T_1, T_2, T_3 + S_3, \ldots, T_{i-2} + S_{i-2}\}, T_{i-1} + S_{i-1}\} \quad (i = 4, \ldots, n). \tag{1}$$

Let us define now the following random variables:

$$M_i = \max\{T_1, T_2, T_3 + S_3, \ldots, T_{i-1} + S_{i-1}\} \quad (i = 2, \ldots, n) \tag{2}$$

with

$$M_2 = T_1 \quad \text{and} \quad M_3 = \max\{T_1, T_2\}.$$

In order to develop recursive relations, we write (2) in the following form:

$$M_i = \max\{\max\{T_1, T_2, T_3 + S_3, \ldots, T_{i-2} + S_{i-2}\}, T_{i-1} + S_{i-1}\} \quad (i = 4, \ldots, n).$$

Then, for the system lifetime, we have:

$$T = \max\{T_1, T_2, T_3 + S_3, \ldots, T_n + S_n\}.$$

With

$$V_i = T_i + S_i \quad (i = 2, \ldots, n), \tag{3}$$

we have

$$S_i = \min\{M_{i-1}, V_{i-1}\} \quad (i = 3, \ldots, n),$$

and

$$M_i = \max\{M_{i-1}, V_{i-1}\} \quad (i = 3, \ldots, n).$$

Consider now the following events:

$$\begin{aligned}
&\text{(i)} && \mathscr{A}_i = \{M_i \leqslant V_i\} \quad (i = 2, \ldots, n-1),\\
&\text{(ii)} && \mathscr{B}_i = \{M_i < c\} \quad (i = 2, \ldots, n),\\
&\text{(iii)} && \mathscr{C}_i = \{V_i < c\} \quad (i = 2, \ldots, n),\\
&\text{(iv)} && \mathscr{D}_i = \mathscr{B}_i \cap \mathscr{C}_i \quad (i = 2, \ldots, n),\\
&\text{(v)} && \mathscr{F}_i = \{T_{i+1} + V_i < c\} \quad (i = 2, \ldots, n-1),\\
&\text{(vi)} && \mathscr{G}_i = \{T_{i+1} + M_i < c\} \quad (i = 2, \ldots, n-1),\\
&\text{(vii)} && \mathscr{Q}_i = \mathscr{F}_i \cap \mathscr{A}'_i \quad (i = 2, \ldots, n-1),\\
&\text{(viii)} && \mathscr{W}_i = \mathscr{G}_i \cap \mathscr{A}_i \quad (i = 2, \ldots, n-1).
\end{aligned} \tag{4}$$

The interpretation of the events in (i)–(vi) is obvious. We only interpret the events $\mathscr{Q}_i$ in (vii) and $\mathscr{W}_i$ in (viii). Both events $\mathscr{Q}_i$ and $\mathscr{W}_i$, refer to the case where the system does not fail before the starting time of the $(i+1)$-unit. This starting time is either the failure time of the latest introduced unit, i.e. that of the i-unit, and thus this unit has to be replaced, event $\mathscr{Q}_i$, or it is the failure time of another one among the $(i-1)$ previously introduced units, and thus that unit has to be replaced, event $\mathscr{W}_i$.

We may notice now the following:

$$\mathscr{D}_n = \mathscr{B}_n \cap \mathscr{C}_n = \{M_n < c\} \cdot \{V_n < c\} = \{\max\{M_n, V_n\} < c\}.$$

Introducing (2) and (3), for $i = n$, we get:

$$\mathscr{D}_n = \{\max\{T_1, T_2, T_3 + S_3, T_4 + S_4, \ldots, T_n + S_n\} < c\} = \{T < c\} = \mathscr{S}',$$

that is $\mathscr{D}_n$, is the complement of the event $\mathscr{S}$ and therefore,

$$P[\mathscr{S}] = 1 - P[\mathscr{D}_n]. \tag{5}$$

The $(n, n-2)$-system reliability is given by the following:

Theorem 1. *The system reliability is*

$$P[\mathscr{S}] = 1 - \sum_{j=1}^{2^{n-2}} \int_{\mathscr{K}_{1j}} \cdots \int_{\mathscr{K}_{nj}} f(\mathbf{t})\, \mathrm{d}t_n \ldots \mathrm{d}t_1$$

with $\mathscr{K}_{ij}$ $(i = 1, \ldots, n;\ j = 1, \ldots, 2^{n-2})$ *given by*:

(i) $\mathscr{K}_{1j} = \{t_1 < c\}$,

(ii) $\mathscr{K}_{ij} = [\{M_i - S_i \leqslant t_i < c - S_i\} \cdot I(\mathscr{Y}_i^{(j)}, \mathscr{W}_i)] \cup [\{0 \leqslant t_i < M_i - S_i\} \cdot I(\mathscr{Y}_i^{(j)}, \mathscr{Q}_i)] \quad (i = 2, \ldots, n-1)$,

(iii) $\mathscr{K}_{nj} = [\{0 \leqslant t_n < c - M_{n-1}\} \cdot I(\mathscr{Y}_{n-1}^{(j)}, \mathscr{W}_{n-1})] \cup [\{0 \leqslant t_n < c - V_{n-1}\} \cdot I(\mathscr{Y}_{n-1}^{(j)}, \mathscr{Q}_{n-1})]$,

where I stands for the indicator function for sets, i.e. $I(\mathscr{Y}, \mathscr{Q}) = \Omega$ *when* $\mathscr{Y} = \mathscr{Q}$ *and* $I(\mathscr{Y}, \mathscr{Q}) = \emptyset$ *when* $\mathscr{Y} \neq \mathscr{Q}$.

The proof is presented in [11] and thus omitted here.

From Theorem 1, we get an exact closed form for the system reliability in cases of iid unit-lifetimes, moderate values of n and mathematically tractable distributions. Theorem 2, gives another formula efficient and easy for simulated evaluation of the system reliability and the MTSF, most useful when dealing with large systems, correlated unit-lifetimes or distributions that do not lead to explicit analytic results. This can also be used as an alternative to Theorem 1 for analytic evaluation of the system reliability. We have to mention here, that the specification of the correlation structure in the joint pdf $f(\mathbf{t})$, is directly related to the system's structure.

Theorem 2. *The system reliability* $P[\mathscr{S}]$ *is given by*

$$P[\mathscr{S}] = 1 - \sum_{j=1}^{2^{n-2}} P\left[\bigcap_{i=1}^{n-1} \mathscr{Y}_i^{(j)}\right],$$

where $\mathscr{Y}_1^{(j)} = \mathscr{D}_2$, *and for i =* $2, \ldots, n-1$, $\mathscr{Y}_i^{(j)}$ *is either* $\mathscr{W}_i = \mathscr{G}_i \cdot \mathscr{A}_i$ *or* $\mathscr{Q}_i = \mathscr{F}_i \cdot \mathscr{A}_i'$ *as j runs from* 1 *to* 2^{n-2} *covering all possible choices.*

Proof. For $i \geqslant 3$, the events $\mathscr{B}_i$ can be written:

$$\mathscr{B}_i = \{M_i < c\} = \{\max\{M_{i-1}, V_{i-1}\} < c\} = \{M_{i-1} < c\} \cdot \{V_{i-1} < c\} = \mathscr{B}_{i-1} \cdot \mathscr{C}_{i-1} = \mathscr{D}_{i-1},$$

and since by definition $\mathcal{D}_i = \mathcal{B}_i \cdot \mathcal{C}_i$ $(i = 2, \ldots, n)$, we have:

$$\mathcal{D}_i = \mathcal{B}_i \cdot \mathcal{C}_i = \mathcal{D}_{i-1} \cdot \mathcal{C}_i \quad (i = 3, \ldots, n) \tag{6}$$

with

$$\mathcal{D}_2 = \mathcal{B}_2 \cdot \mathcal{C}_2 = \{T_1 < c\} \cdot \{T_2 < c\}. \tag{7}$$

We may analyse now the event $\mathcal{C}_i$ $(i = 3, \ldots, n)$, as a union of two incompatible events. Specifically

$$\mathcal{C}_i = \mathcal{C}_i \cdot \mathcal{A}_{i-1} \cup \mathcal{C}_i \cdot \mathcal{A}'_{i-1} \quad (i = 3, \ldots, n), \tag{8}$$

and since $S_2 = 0$, we have also

$$\mathcal{C}_2 = \{T_2 + S_2 < c\} = \{T_2 < c\}.$$

Since,

$$\begin{aligned}\mathcal{C}_i \cdot \mathcal{A}_{i-1} &= \{T_i + \min\{M_{i-1}, V_{i-1}\} < c\} \cdot \{M_{i-1} \leqslant V_{i-1}\} = \{T_i + M_{i-1} < c\} \cdot \{M_{i-1} \leqslant V_{i-1}\} \\ &= \mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1} \quad (i = 3, \ldots, n),\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_i \cdot \mathcal{A}'_{i-1} &= \{T_i + \min\{M_{i-1}, V_{i-1}\} < c\} \cdot \{M_{i-1} > V_{i-1}\} = \{T_i + V_{i-1} < c\} \cdot \{M_{i-1} > V_{i-1}\} \\ &= \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1} \quad (i = 3, \ldots, n),\end{aligned}$$

the relation (8), can be written:

$$\mathcal{C}_i = \mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1} \cup \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1} \quad (i = 3, \ldots, n),$$

and the result (6), takes the following recursive form:

$$\mathcal{D}_i = (\mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1} \cup \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1}) \cdot \mathcal{D}_{i-1} \quad (i = 3, \ldots, n) \tag{9}$$

with $\mathcal{G}_i$, $\mathcal{F}_i$, and $\mathcal{A}_i$ given by (4). Thus, for $i = n$, the above recursive relation gives:

$$\mathcal{D}_n = \left\{ \bigcap_{i=3}^{n} (\mathcal{G}_{i-1} \cdot \mathcal{A}_{i-1} \cup \mathcal{F}_{i-1} \cdot \mathcal{A}'_{i-1}) \right\} \cap \mathcal{D}_2. \tag{10}$$

Let now $\mathcal{W}_i = \mathcal{G}_i \cdot \mathcal{A}_i$ and $\mathcal{Q}_i = \mathcal{F}_i \cdot \mathcal{A}'_i$ $(i = 2, \ldots, n-1)$. Since the events $\mathcal{W}_i$ and $\mathcal{Q}_i$ are incompatible, the probability of the event $\mathcal{D}_n$ can be written:

$$P[\mathcal{D}_n] = P\left[\left\{ \bigcap_{i=2}^{n-1} (\mathcal{G}_i \cdot \mathcal{A}_i \cup \mathcal{F}_i \cdot \mathcal{A}'_i) \right\} \cdot \mathcal{D}_2 \right] = P\left[\left\{ \bigcap_{i=2}^{n-1} (\mathcal{W}_i \cup \mathcal{Q}_i) \right\} \cdot \mathcal{D}_2 \right] = P\left[\bigcup_{j=1}^{2^{n-2}} \bigcap_{i=1}^{n-1} \mathcal{Y}_i^{(j)} \right]$$

or

$$P[\mathcal{D}_n] = \sum_{j=1}^{2^{n-2}} P\left[\bigcap_{i=1}^{n-1} \mathcal{Y}_i^{(j)} \right], \tag{11}$$

where $\mathcal{Y}_1^{(j)} = \mathcal{D}_2$, and for $i = 2, \ldots, n-1$, $\mathcal{Y}_i^{(j)}$ is either $\mathcal{W}_i = \mathcal{G}_i \cdot \mathcal{A}_i$ or $\mathcal{Q}_i = \mathcal{F}_i \cdot \mathcal{A}'_i$, as j runs from 1 to 2^{n-2} covering all possible choices. □

4. Applications

In this section, we give the final expressions for the system reliability, applying Theorem 2 for $n = 3$, 4 and 5. These expressions are used in Sections 5 and 6 for simulation purposes. They can also be used for exact analytic evaluations and comparisons.

4.1. (2, 1)-system (n = 3)

For $n = 3$, we have from (5)

$$P[\mathscr{S}] = P[T \geqslant c] = 1 - P[\mathscr{D}_3]$$

with $P[\mathscr{D}_3]$ given by (11), namely

$$P[\mathscr{D}_3] = \sum_{j=1}^{2} P\left[\bigcap_{i=1}^{2} \mathscr{Y}_i^{(j)}\right],$$

where $\mathscr{Y}_1^{(j)} = \mathscr{D}_2$ and $\mathscr{Y}_2^{(j)}$ is $\mathscr{W}_2 = \mathscr{G}_2 \cdot \mathscr{A}_2$, for $j = 1$, and $\mathscr{Q}_2 = \mathscr{F}_2 \cdot \mathscr{A}'_2$, for $j = 2$.

Thus,

$$P[\mathscr{D}_3] = P[\mathscr{D}_2\mathscr{W}_2] + P[\mathscr{D}_2\mathscr{Q}_2].$$

From the relations in (4), we get

$$\mathscr{A}_2 = \{T_1 \leqslant T_2\}, \quad \mathscr{G}_2 = \{T_3 + T_1 < c\}, \quad \text{and} \quad \mathscr{F}_2 = \{T_3 + T_2 < c\}.$$

Then we have:

$$\mathscr{W}_2 = \mathscr{G}_2 \cdot \mathscr{A}_2 = \{T_3 + T_1 < c\} \cdot \{T_1 \leqslant T_2\}, \tag{12}$$

and

$$\mathscr{Q}_2 = \mathscr{F}_2 \cdot \mathscr{A}'_2 = \{T_3 + T_2 < c\} \cdot \{T_1 > T_2\}. \tag{13}$$

Using the above two relations and (7), the probability of the event $\mathscr{D}_3$ is:

$$P[\mathscr{D}_3] = P[\{T_2 < c\} \cdot \{T_3 + T_1 < c\} \cdot \{T_1 \leqslant T_2\}] + P[\{T_1 < c\} \cdot \{T_3 + T_2 < c\} \cdot \{T_1 > T_2\}],$$

and therefore, we have:

$$P[\mathscr{S}] = 1 - P[\{T_2 < c\} \cdot \{T_3 + T_1 < c\} \cdot \{T_1 \leqslant T_2\}] - P[\{T_1 < c\} \cdot \{T_3 + T_2 < c\} \cdot \{T_1 > T_2\}]. \tag{14}$$

4.2. (2, 2)-system (n = 4)

For $n = 4$, we have

$$P[\mathscr{S}] = P[T \geqslant c] = 1 - P[\mathscr{D}_4]$$

with $P[\mathscr{D}_4]$ given by (11), namely

$$P[\mathscr{D}_4] = \sum_{j=1}^{4} P\left[\bigcap_{i=1}^{3} \mathscr{Y}_i^{(j)}\right],$$

where $\mathscr{Y}_1^{(j)} = \mathscr{D}_2$ and for $i = 2, 3$, $\mathscr{Y}_i^{(j)}$ is either $\mathscr{W}_i$ or $\mathscr{Q}_i$, as j runs from 1 to 4.

Thus,

$$P[\mathscr{D}_4] = P[\mathscr{D}_2\mathscr{W}_2\mathscr{W}_3] + P[\mathscr{D}_2\mathscr{Q}_2\mathscr{W}_3] + P[\mathscr{D}_2\mathscr{W}_2\mathscr{Q}_3] + P[\mathscr{D}_2\mathscr{Q}_2\mathscr{Q}_3].$$

Since

$$M_3 = \max\{T_1, T_2\}, \quad S_3 = \min\{T_1, T_2\},$$

the relations (4i), (4vi), (4v) for $i = 3$, can be written respectively:

$$\mathscr{A}_3 = \{\max\{T_1, T_2\} \leqslant T_3 + \min\{T_1, T_2\}\}, \quad \mathscr{G}_3 = \{T_4 + \max\{T_1, T_2\} < c\},$$

and

$$\mathscr{F}_3 = \{T_4 + T_3 + \min\{T_1, T_2\} < c\}.$$

Then, we have

$$\mathscr{W}_3 = \mathscr{G}_3 \cdot \mathscr{A}_3 = \{T_4 + \max\{T_1, T_2\} < c\} \cdot \{\max\{T_1, T_2\} \leqslant T_3 + \min\{T_1, T_2\}\}, \tag{15}$$

and

$$\mathscr{Q}_3 = \mathscr{F}_3 \cdot \mathscr{A}'_3 = \{T_4 + T_3 + \min\{T_1, T_2\} < c\} \cdot \{\max\{T_1, T_2\} > T_3 + \min\{T_1, T_2\}\}. \tag{16}$$

Using the relations (7), (12) and (13) together with the above two relations, the probability of the event $\mathscr{S}$, is given by:

$$\begin{aligned} P[S] = 1 &- P[\{T_3 + T_1 < c\} \cdot \{T_4 + T_2 < c\} \cdot \{T_2 \leqslant T_3 + T_1\} \cdot \{T_1 \leqslant T_2\}] - P[\{T_3 + T_2 < c\} \\ &\cdot \{T_4 + T_1 < c\} \cdot \{T_1 \leqslant T_3 + \mathrm{T}_2\} \cdot \{T_1 > T_2\}] - P[\{T_2 < c\} \cdot \{T_4 + T_3 + T_1 < c\} \\ &\cdot \{T_2 > T_3 + T_1\} \cdot \{T_1 \leqslant T_2\}] - P[\{T_1 < c\} \cdot \{T_4 + T_3 + T_2 < c\} \cdot \{T_1 > T_3 + T_2\} \cdot \{T_1 > T_2\}]. \end{aligned} \tag{17}$$

4.3. (2, 3)-system (n = 5)

For $n = 5$, we have

$$P[\mathscr{S}] = P[T \geqslant c] = 1 - P[\mathscr{D}_5].$$

Similarly, the probability of the event $\mathscr{S}$, is given by:

$$\begin{aligned} P[\mathscr{S}] = 1 &- P[T_2 < c, \quad T_1 \leqslant T_2, \quad T_3 + T_1 < c, \quad T_2 \leqslant T_3 + T_1, \quad T_4 + T_2 < c, \\ &T_3 + T_1 < T_4 + T_2, \quad T_5 + T_3 + T_1 < c] \\ &- P[T_1 < c, \quad T_1 > T_2, \quad T_3 + \mathrm{T}_2 < c, \quad T_1 \leqslant T_3 + T_2, \quad T_4 + T_1 < c, \quad T_3 + T_2 < T_4 + T_1, \\ &T_5 + T_3 + T_2 < c] - P[T_2 < c, \quad T_1 \leqslant T_2, \quad T_3 + T_1 < c, \quad T_2 > T_3 + T_1, \\ &T_4 + T_3 + T_1 < c, \quad T_2 < T_4 + T_3 + T_1, \quad T_5 + T_2 < c] - P[T_1 < c, \quad T_1 > T_2, \\ &T_3 + T_2 < c, \quad T_1 > T_3 + T_2, \quad T_4 + T_3 + T_2 < c, \quad T_1 < T_4 + T_3 + T_2, \quad T_5 + T_1 < c] \\ &- P[T_2 < c, \quad T_1 \leqslant T_2, \quad T_3 + T_1 < c, \quad T_2 \leqslant T_3 + T_1, \quad T_4 + T_2 < c, \quad T_3 + T_1 > T_4 + T_2, \\ &T_5 + T_4 + T_2 < c] - P[T_1 < c, \quad T_1 > T_2, \quad T_3 + T_2 < c, \quad T_1 \leqslant T_3 + T_2, \quad T_4 + T_1 < c, \\ &T_3 + T_2 > T_4 + T_1, \quad T_5 + T_4 + T_1 < c] - P[T_2 < c, \quad T_1 \leqslant T_2, \quad T_3 + T_1 < c, \\ &T_2 > T_3 + T_1, \quad T_4 + T_3 + T_1 < c, \quad T_2 > T_4 + T_3 + T_1, \quad T_5 + T_4 + T_3 + T_1 < c] \\ &- P[T_1 < c, \quad T_1 > T_2, \quad T_3 + T_2 < c, \quad T_1 > T_3 + T_2, \quad T_4 + T_3 + T_2 < c, \\ &T_1 > T_4 + T_3 + T_2, \quad T_5 + T_4 + T_3 + T_2 < c]. \end{aligned} \tag{18}$$

5. Analytic and simulated results with independent unit-lifetimes and special distributions

Based on Theorem 2, we developed simulation programs which compute the system reliability and the MTSF, for either independent or correlated unit-lifetimes. In this section, we consider the case of an identical $(2, n-2)$-system with independent unit-lifetimes exponentially distributed for $n = 3, 4$ and 5, and the case of a non-identical three-unit system with independent unit-lifetimes exponentially distributed. We also consider an identical three-unit system with independent unit-lifetimes following a special Weibull distribution. The reliability for all the above systems has been provided in exact closed form presented in [11]. Analytic results for the system reliability and MTSF, have been derived and comparison with the simulated ones are provided.

All simulations programs have developed in the *Mathematica* environment and the simulated values have resulted after 50,000 iterations. The algorithm of the simulation programs is presented in pseudo-code in Appendix I.

5.1. $(2, n-2)$-identical system with exponential lifetimes $(n = 3, 4, 5)$

In [11], the reliability $P[\mathscr{S}]$, is derived from Theorem 1, for a n-identical unit system $(n = 3, 4, 5)$, with unit-lifetimes independent and exponentially distributed with parameter b. Specifically, we have:

$$P[\mathscr{S}] = P[T \geqslant c] = -3e^{-2bc} - 2bce^{-2bc} + 4e^{-bc} \quad (n = 3), \tag{19}$$

$$P[\mathscr{S}] = P[T \geqslant c] = -7e^{-2bc} - 6bce^{-2bc} - 2b^2c^2e^{-2bc} + 8e^{-bc} \quad (n = 4), \tag{20}$$

$$P[\mathscr{S}] = P[T \geqslant c] = -15e^{-2bc} - 14bce^{-2bc} - 6b^2c^2e^{-2bc} - \frac{4}{3}b^3c^3e^{-2bc} + 16e^{-bc}(n = 5). \tag{21}$$

Applying the above results for $b = 0.04$, i.e. with $E[T_i] = 25$ $(i = 1, \ldots, n)$, and $c = 10, 20, \ldots, 60$, we get the exact values for $P[\mathscr{S}]$. These are presented in the second column of Tables 1–3. Using the results (14), (17) and (18) for $n = 3, 4$ and 5 respectively, and running the corresponding simulation programs for the same values of b and c, we get the numerical estimates of the reliability $P[\mathscr{S}]$ and the MTSFs presented in the third and fifth column of Tables 1–3 respectively. In these tables, it is also presented the exact MTSFs, as they can be derived from (19)–(21), using the relation $\text{MTSF} = \int_0^\infty R(t)\,dt$.

Remark 1. We have investigated the problem of approximating the pdf of the above $(2, n-2)$-system lifetime by an appropriately chosen Weibull pdf. There are several ways to choose an approximating Weibull. We have investigated the following approximations: (i) by equating modes and curvature at modes, (ii) by equating means and variances and (iii) by equating modes and variances. We have found out that the best approximation was that provided by (iii). As an example, we present the approximation of the reliability function (19), when $b = 0.04$. The values of the Weibull parameters (α, β) are respectively: (i)

Table 1
Exponentially iid $(n = 3)$

c	$P[\mathscr{S}]$		MTSF	
	Exact	Estimated	Exact	Estimated
10	0.9738	0.9736		
20	0.8686	0.8691		
30	0.7149	0.7142	50.00	49.95
40	0.5549	0.5542		
50	0.4131	0.4159		
60	0.2987	0.2985		

Table 2
Exponentially iid ($n = 4$)

c	$P[\mathscr{S}]$		MTSF	
	Exact	Estimated	Exact	Estimated
10	0.9951	0.9950		
20	0.9538	0.9542		
30	0.8601	0.8609	62.50	62.45
40	0.7298	0.7297		
50	0.5882	0.5883		
60	0.4548	0.4530		

Table 3
Exponentially iid ($n = 5$)

c	$P[\mathscr{S}]$		MTSF	
	Exact	Estimated	Exact	Estimated
10	0.9992	0.9991		
20	0.9865	0.9877		
30	0.9415	0.9418	75.00	74.99
40	0.8571	0.8496		
50	0.7429	0.7447		
60	0.6154	0.6110		

$(1.9669, 45.0694)$, (ii) $(1.6790, 55.9893)$ and (iii) $(1.6525, 55.1175)$. Figs. 2a–c present the corresponding approximations of the reliability function (19), for the above three cases. It seems that the Weibull approximation would be better for large systems, but this needs a more thorough investigation. The study of the asymptotic behavior of the magnitude of error would be also of interest.

5.2. (2, 1)-non-identical system with exponential lifetimes ($n = 3$)

In the case of a three-unit system with unit-lifetimes independent and exponentially distributed with different parameters b_i ($i = 1, 2, 3$), the exact expression of the system reliability $P[\mathscr{S}]$, according to [11], is quite large and thus is presented in Appendix II.

Applying the result in Appendix II, for $b_1 = 0.05$, $b_2 = 0.04$, and $b_3 = 0.025$, i.e. with $E[T_1] = 20$, $E[T_2] = 25$ and $E[T_3] = 12.5$ and $c = 10, 20, \ldots, 60$, and running the simulation program that makes use of the result (14) for the same values of b's and c, we get numerical results for the reliability $P[\mathscr{S}]$ and MTSF, presented in Table 4.

Remark 2. It is interesting to notice that reliability function in Appendix II is symmetrical with respect to b_1 and b_2. This is also true with any lifetime distributions since the first two units start simultaneously. With different lifetime distributions, it might be of interest to optimize the system with respect to the order that the units are introduced. In the case of three non-identical units with exponential lifetimes, it can be simply proved, from the reliability function in Appendix II, that the probability of successful control is maximized when the best unit, i.e. the one with the largest mean lifetime, is kept as standby. For example, with $c = 10$ and (b_1, b_2, b_3), the permutations of the numbers 0.05, 0.04, and 0.025, we obtain as the first row of the Tables 4 and 5 shows, $P[\mathscr{S}] = 0.9793$ when $b_3 = 0.025$, and $P[\mathscr{S}] = 0.9790$ when $b_3 = 0.05$, while the interchange of b_1 with b_2 does not matter. Comparing the corresponding MTSFs, the effect of the ordering

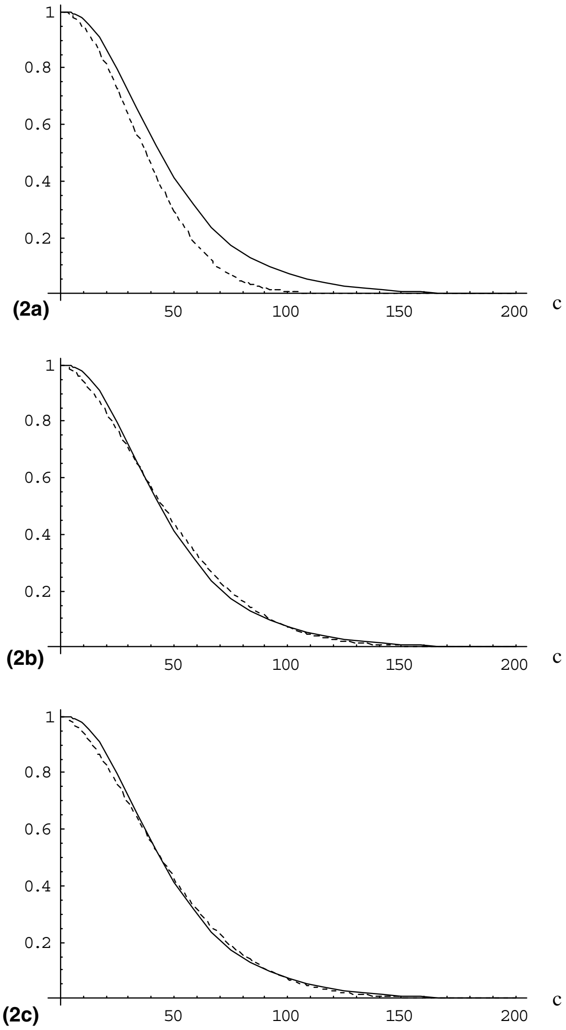

Fig. 2. Reliability function for required period c of a three identical unit system with independent unit-lifetimes exponentially distributed ($b = 0.04$). The dotted line corresponds to the Weibull approximation.

Table 4
Optimal ordering of units ($n = 3$)

c	$P[\mathscr{S}]$		MTSF	
	Exact	Estimated	Exact	Estimated
10	0.9793	0.9791		
20	0.8942	0.8945		
30	0.7661	0.7660	59.38	59.40
40	0.6276	0.6267		
50	0.4992	0.5003		
60	0.3897	0.3870		

is much more apparent. Analytic results in the case of a non-identical four-unit system with exponential lifetimes have been also derived. Numerical examples show again that the better unit has to be used later.

Table 5
Random ordering of units ($n = 3$)

c	$P[\mathscr{S}]$		MTSF	
	Exact	Estimated	Exact	Estimated
10	0.9790	0.9788		
20	0.8917	0.8923		
30	0.7591	0.7607	57.14	57.15
40	0.6152	0.6168		
50	0.4821	0.4843		
60	0.3694	0.3686		

Table 6
Weibull iid ($n = 3$)

c	$P[\mathscr{S}]$		MTSF	
	Exact	Estimated	Exact	Estimated
0.3	0.9998	0.9998		
0.5	0.9967	0.9967		
0.7	0.9802	0.9800	1.61	1.61
0.9	0.9330	0.9331		
1.1	0.8418	0.8410		
1.3	0.7093	0.7089		

5.3. (2, 1)-identical system with Weibull lifetimes (n = 3)

In the case of a three-unit system with independent unit-lifetimes following a special Weibull distribution with parameters $b = 2$ and $\lambda = 1$, i.e. when $f_i(t) = 2t\mathrm{e}^{-t^2}$, $E[T_i] = \sqrt{\pi}/2$ ($i = 1, 2, 3$), then according to [11], the reliability of the system $P[\mathscr{S}]$, is given by:

$$P[\mathscr{S}] = P[T \geqslant c]$$
$$= -\frac{5}{3}\mathrm{e}^{-2c^2} + \frac{8}{3}\mathrm{e}^{-c^2} - c\mathrm{e}^{-\frac{3}{2}c^2}\sqrt{2\pi}\mathrm{Erf}\left[\frac{c}{\sqrt{2}}\right] + \frac{2}{3}c\mathrm{e}^{-\frac{2}{3}c^2}\sqrt{\frac{\pi}{3}}\mathrm{Erf}\left[\frac{c}{\sqrt{3}}\right] + \frac{2}{3}c\mathrm{e}^{-\frac{2}{3}c^2}\sqrt{\frac{\pi}{3}}\mathrm{Erf}\left[\frac{2c}{\sqrt{3}}\right], \qquad (22)$$

where

$$\mathrm{Erf}[z] = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

Applying the result (22) and running the simulation program we get numerical results for the reliability $P[\mathscr{S}]$ and the MTSF, which are presented in Table 6.

6. Simulated results with special distributions

From the comparisons made in the previous section, we have realized that in all cases the simulated results come very close to the exact values of the system reliability and the MTSF. Thus, the simulation method applied is quite good for most realistic situations. For large systems or for less mathematically tractable distributions, even in the case of independent unit-lifetimes, the simulation approach is appropriate. In the following subsections we present simulated values for identical $(2, n-2)$-system ($n = 4, 5$) with independent unit-lifetimes following a certain Weibull distribution. We present also a non-identical

$(2, n-2)$-system ($n = 3, 4$) with independent unit-lifetimes following Weibull distributions. Finally, for the case of correlated lifetimes we present a two-unit system where the joint distribution of the lifetimes of both units is taken to be the bivariate exponential. The same procedure can be used for larger number of units provided that the correlation structure of the units is specified.

6.1. $(2, n-2)$-identical system with Weibull lifetimes ($n = 4, 5$)

In a $(2, n-2)$-system with independent unit-lifetimes following a special Weibull distribution with parameters $b = 2$ and $\lambda = 1$, i.e. when $f_i(t) = 2te^{-t^2}$ ($i = 1, \ldots, n$; $n = 4, 5$), the simulated results for the probability $P[\mathscr{S}]$ and the MTSF, are presented respectively in Tables 7 and 8.

6.2. $(2, n-2)$-non-identical system with Weibull lifetimes ($n = 3, 4$)

As another example, we present in Table 9, the simulated values for $P[\mathscr{S}]$ and MTSF, of a three-unit system with independent unit-lifetimes following Weibull distributions with different parameters. Taking $b_1 = 2$, $b_2 = 3$, $b_3 = 4$ and $\lambda = 1$ we obtain:

Table 7
Weibull iid ($n = 4$)

c	Estimated $P[\mathscr{S}]$	Estimated MTSF
0.8	0.9962	
1.1	0.9710	
1.4	0.8873	2.05
1.7	0.7215	
2.0	0.5135	
2.3	0.3154	

Table 8
Weibull iid ($n = 5$)

c	Estimated $P[\mathscr{S}]$	Estimated MTSF
0.8	0.9998	
1.1	0.9962	
1.4	0.9775	2.52
1.7	0.9158	
2.0	0.7838	
2.3	0.6000	

Table 9
Non-identical units with Weibull lifetimes ($n = 3$)

c	Estimated $P[\mathscr{S}]$	Estimated MTSF
0.5	0.9998	
0.8	0.9915	
1.1	0.9208	1.61
1.4	0.6998	
1.7	0.3938	
2.0	0.1534	

We consider also a four-unit system with independent unit-lifetimes following Weibull distributions with different parameters b_i $(i = 1, \dots, 4)$ and $\lambda = 1$. Then the simulated values for $P[\mathscr{S}]$ and MTSF, when $b_1 = 2$, $b_2 = 3$, $b_3 = 4$ and $b_4 = 5$, are presented in Table 10.

6.3. Two-unit (2, 0)-system with correlated unit-lifetimes following a bivariate exponential distribution

The lifetimes T_1 and T_2, are assumed here to have the bivariate exponential distribution. Specifically, the joint pdf is:

$$f(t_1, t_2) = \lambda\mu(1 - r)e^{-(\lambda t_1 + \mu t_2)} I_0(2(\lambda\mu r t_1 t_2)^{1/2}), \quad \lambda, \mu, t_1, t_2 > 0, \ |r| < 1, \tag{23}$$

where, cf. Gupta et al. [7],

$$I_0(z) = \sum_{k=0}^{\infty} (z/2)^{2k} / (k!)^2,$$

the Bessel function of type I and order zero,

$$I_0(2(\lambda\mu r t_1 t_2)^{1/2}) = \sum_{k=0}^{\infty} \frac{(\lambda\mu r t_1 t_2)^k}{(k!)^2}.$$

Thus, the marginal probability distribution function of T_1, equals:

$$f(t_1) = \lambda(1 - r)e^{-\lambda(1-r)t_1}, \quad \lambda > 0, \ t_1 > 0, \ |r| < 1, \tag{24}$$

and the conditional probability distribution function of $T_2 | T_1 = t_1$, equals:

$$f(t_2 | T_1 = t_1) = \mu e^{-(\lambda r t_1 + \mu t_2)} I_0(2(\lambda\mu r t_1 t_2)^{1/2}), \quad t_2 > 0, \ \lambda, \mu, t_1 > 0, \ |r| < 1. \tag{25}$$

The simulation results when $\lambda = 0.5$, $\mu = 0.3$ and $r = 0.4$ are presented in Table 11.

Remark 3. In order to use the Theorem 2, we generate correlated random pairs (T_1, T_2) that have the joint pdf (23), as follows. T_1, is generated according to (24), and T_2, according to (25), using the rejection

Table 10
Non-identical units with Weibull lifetimes ($n = 4$)

c	Estimated $P[\mathscr{S}]$	Estimated MTSF
0.8	0.9999	
1.1	0.9979	
1.4	0.9650	2.03
1.7	0.8139	
2.0	0.5158	
2.3	0.2253	

Table 11
Correlated unit-lifetimes ($n = 2$)

c	Estimated $P[\mathscr{S}]$	Estimated MTSF
2	0.817	
4	0.596	6.39
6	0.409	

method. The same procedure can be used for larger values of units provided that we specify the joint pdf of the non-failed unit and the new entering unit. The applied rejection method has worked quite well at a reasonable length of time. Alternative rejection methods together with an extensive presentation of sampling algorithms can be found in Ripley [12].

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Appendix I

All simulation programs for the computations of the system reliability and the MTSF, have been developed in the *Mathematica* environment, and ran in a Pentium IV computer. The simulation values for independent unit-lifetimes have been derived after 50,000 iterations. In the case of correlated unit-lifetimes the rejection method has been applied and the simulation values have been derived after 1000 iterations. In the independence case, the production of each one of Tables 1–10 required about 2 h running time, while in case of correlated unit-lifetimes the production of Table 11 required about 7 h.

The algorithm of the simulation programs, presented in pseudo-code, is given below:

```
begin
  read n;   (*Number of units*)
  read c1;  (*Initial*)
  read k;   (*Step*)
  read c2;  (*Final*)
  read r;   (*Iterations*);
  for c = c1 to c2 step k
    i = 0;
    j = 0;
    array M[r];
    l = 1;
    for w = 1 to r
      array K[n];
      K[n] = call random numbers( );   (*In case of correlated lifetimes the rejection method is used*)
      if relation 11 is true, then i = i + 1, else j = j + 1;
      T = f(K[n]);   (*Evaluate from the unit-lifetimes the system-lifetime T using relation 11*)
      M[l] = T;
      l = l + 1;
    end for;
    R = i/r;
    Print "When c=" c "the system reliability R=" R;
    MTSF = (Sum_{l=1}^{r} M[l]) / r;
    Print "The mean time to system failure is" MTSF;
  end for;
end
```

Appendix II

Papageorgiou and Kokolakis [11], in the case of a three-unit system with independent lifetimes following Exponential distributions with parameters b_i $(i = 1, 2, 3)$, obtain the following result for the system reliability $P[\mathscr{S}]$:

$$
\begin{aligned}
P[\mathscr{S}] = P[T \geqslant c] = 1 &- \frac{b_1^3}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{b_2^3}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&- \frac{b_1^2 b_2}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{b_2^2 b_1}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{2b_1^2 b_3}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{2b_2^2 b_3}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{b_1 b_2 b_3}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{b_1 b_2 b_3}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&- \frac{b_1 b_3^2}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{b_2 b_3^2}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{b_2^2 b_1 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{b_1^2 b_2 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{2b_1^2 b_2 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{2b_1 b_2^2 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{b_1^3 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{b_2^3 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&- \frac{b_2^2 b_3 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{b_1^2 b_3 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&- \frac{3b_1 b_2 b_3 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{3b_1 b_2 b_3 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&- \frac{2b_1^2 b_3 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{2b_2^2 b_3 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{b_2 b_3^2 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{b_1 b_3^2 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{b_1 b_3^2 e^{-b_2 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{b_2 b_3^2 e^{-b_1 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{b_2^2 b_3 e^{-(b_1+b_2)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{b_1^2 b_3 e^{-(b_1+b_2)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&+ \frac{2b_1 b_2 b_3 e^{-(b_1+b_2)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} + \frac{2b_1 b_2 b_3 e^{-(b_1+b_2)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&- \frac{b_2 b_3^2 e^{-(b_1+b_2)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{b_1 b_3^2 e^{-(b_1+b_2)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)} \\
&- \frac{b_1^3 e^{-b_3 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)} - \frac{b_2^3 e^{-b_3 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{b_1^2 b_2 e^{-b_3 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}+\frac{b_1 b_2^2 e^{-b_3 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&-\frac{b_1^2 b_3 e^{-b_3 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}-\frac{b_2^2 b_3 e^{-b_3 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&-\frac{b_1 b_2 b_2 e^{-b_3 c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}-\frac{b_1 b_2 b_2 e^{-b_3 c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&-\frac{b_1^3 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}-\frac{b_2^3 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&-\frac{2b_1^2 b_2 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}-\frac{2b_1 b_2^2 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&-\frac{b_1 b_2^2 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}-\frac{b_1^2 b_2 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&+\frac{b_1^2 b_3 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}+\frac{b_2^2 b_3 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}\\
&+\frac{b_1 b_2 b_3 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_1-b_3)(b_1+b_2-b_3)}+\frac{b_1 b_2 b_3 e^{-(b_2+b_3)c}}{(b_1+b_2)(b_2-b_3)(b_1+b_2-b_3)}.
\end{aligned}
$$

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