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Scheduling starting times for an active redundant system with non-identical lifetimes

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Abstract

Here we examine an active redundant system with scheduled starting times of the units. We assume availability of n non-identical, non-repairable units for replacement or support. The original unit starts its operation at time $s_1 = 0$ and each one of the $(n - 1)$ standbys starts its operation at scheduled time s_i $(i = 2,\ldots,n)$ and works in parallel with those already introduced and not failed before s_i. The system is up at times s_i $(i = 2,\ldots,n)$, if and only if, there is at least one unit in operation. Thus, the system has the possibility to work with up to n units, in parallel structure. Unit-lifetimes T_i $(i = 1,\ldots,n)$ are independent with cdf F_i, respectively. The system has to operate without inspection for a fixed period of time c and it stops functioning when all available units fail before c. The probability that the system is functioning for the required period of time c depends on the distribution of the unit-lifetimes and on the scheduling of the starting times s_i. The reliability of the system is evaluated via a recursive relation as a function of the starting times s_i $(i = 2,\ldots,n)$. Maximizing with respect to the starting times we get the optimal ones. Analytical results are presented for some special distributions and moderate values of n.

Keywords: Active redundancy; Non-identical units; Optimal ordering; Parallel system; Recursive probabilistic analysis; Reliability; Standbys

1. Introduction

Repairable two-unit redundant systems have attracted the attention of several researchers in the field of reliability theory. Many workers, including Murari and Goel [1], Gupta and Goel [2], Goel et al. [3] and Gupta and Chaudhary [4], have investigated the two-unit standby system models assuming that, a unit is replaced instantaneously at its failure by a standby one. Gupta and Kishan [5] consider situations of two-unit system where the standby unit does not operate instantaneously but a fixed preparation time is required to put standby and repaired units into operation. In all these system models it is assumed that the lifetimes are uncorrelated random variables. Papageorgiou and Kokolakis [6] have investigated a related problem where a

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Nomenclature

U_i	the ith unit introduced into the system
T_i	lifetime of ith unit
$f_i(\cdot)$, $F_i(\cdot)$	PDF and CDF of the ith unit-lifetime
$R_i(\cdot)$	reliability function of ith unit
s_i	starting time of the ith unit
n	number of available non repairable units
T	system lifetime
MTSF	mean time to system failure
c	fixed required period of system operation
$\mathcal{S}$	the event $\{T \geqslant c\}$

two-unit general parallel system is supported by $(n - 2)$ cold standby units. It is a case of a passive redundant system where two units start their operation simultaneously at time $s = 0$ and one of these is replaced by a new one upon its failure. Thus, the starting times s_i are random. The main result there, is the evaluation of the system reliability by using recursive probabilistic analysis for non-independent unit lifetimes with general distributions, unlike most earlier results which refer to specific unit lifetime distributions.

The problem of where to allocate a redundant unit in a system in order to optimize the system-lifetime is an important problem in reliability theory. Many workers including El-Neweihi and Sethuraman [7], El-Neweihi et al. [8], Shaked and Shanthikumar [9], and Singh and Misra [10], considered the above problem. There are two common forms of redundancy in reliability theory, namely active and standby redundancy, according to whether a spare unit starts its operation simultaneously with an original unit or at its failure time. Boland et al. [11], considered the problem of allocating a redundant spare in a k-out-of-n system, where the units are independent and stochastically ordered and showed that for active redundancy it is stochastically optimal always to allocate the redundant unit to the weakest component. In the case of standby redundancy, sufficient conditions were also provided for series and parallel systems in order to stochastically optimize the lifetime of such systems. Mi [12], considered a related problem where there are $n + r$ $(1 \leqslant r \leqslant n)$ available components. The problem of which r components have to be used for active redundancy, and where to allocate them in order to maximize the lifetime of the resulting k-out-of-n system was studied.

Here, we examine a related problem. It refers to an n-active redundant system where the main difference from the above is that it has to work without inspection. Thus, the starting times s_i have to be scheduled. In order to maximize the system reliability we require the spare $(n - 1)$ units to start their operation at optimal scheduled times. It is a case of an active redundancy in that a position is functioning if either the original unit or a spare one is functioning. Unlike most earlier studies, the problem here is, not where to allocate a spare unit but when to invoke it in order to optimize the system reliability. The n units are non-repairable and non-identical. The first unit starts its operation at time $s_1 = 0$ and the rest $(n - 1)$ are standbys. Each one of the $(n - 1)$ standbys starts its operation at scheduled time s_i $(i = 2,\ldots,n)$ and works in parallel with those already introduced and not failed before s_i. In Section 3, the system reliability is evaluated via recursive relation as a function of the starting times s_i $(i = 2,\ldots,n)$. In Section 4, applying this recursive relation we get the final expression for the system reliability in closed form in terms of the unit's reliability functions $R_i's$ $(i = 1,\ldots,n)$. In Section 5, introducing into this closed form some special lifetime-distributions and maximizing with respect to the $s_i's$ we get the optimal starting times. The mean time to system failure (MTSF) is evaluated in the same section. Comparisons with the applied policy in Papageorgiou and Kokolakis [6] are made. Optimal ordering of non-identical units is also considered.

2. System description

The problem of the successful control of a process by an active redundant system with scheduled staring times of the units, is considered in this paper.

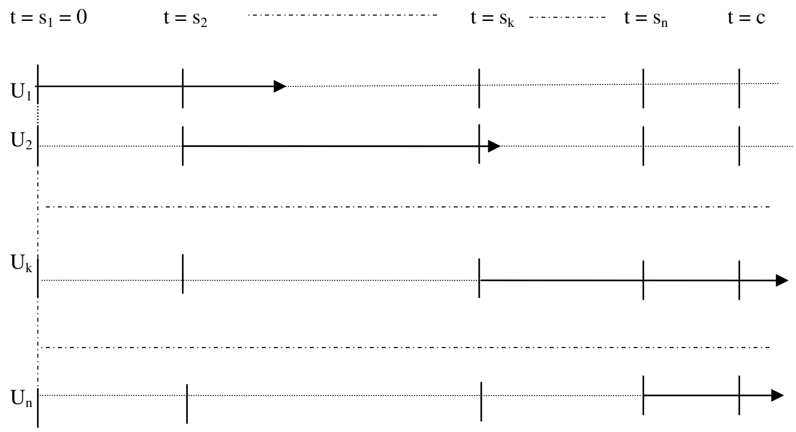

Fig. 1. Units $U_1, U_2, \ldots, U_n$ starting operation at scheduled times $s_1, s_2, \ldots, s_n$, respectively.

Here, there is available a fixed number of n non-repairable and non-identical units. The unit-lifetimes T_i ($i = 1, \ldots, n$) are random independently distributed with cdf F_i, respectively. The process is considered to have a fixed duration c. The control of the process is considered successful provided that during the required period of time c, at least one unit is in operation and the process information required for the control is transferred instantaneously to a new entering unit from a working one. The process is initially controlled by one unit and the remaining $(n - 1)$ units are standbys. The initial unit U_1 starts its operation at time $s_1 = 0$. The second unit U_2 starts its operation at scheduled time $s_2 \geqslant s_1 = 0$, provided that the first unit was in operation up to time s_2 after which both units work in parallel. Similarly, the third unit U_3 starts its operation at scheduled time $s_3 \geqslant s_2$ and works in parallel with those that were in operation up to its entrance time s_3.

In general, the ith unit starts its operation at time $s_i \geqslant s_{i-1}, \ldots, \geqslant s_2 \geqslant s_1$, and it works in parallel with those already introduced and not failed before s_i (see Fig. 1). Thus, the process is simultaneously controlled by at most $(n - 1)$ working units until the entrance of the last available one and it stops functioning when all units have failed. The probability of the successful control of the process till its completion time c depends on the distributions of unit-lifetimes and their starting times s_i ($i = 2, \ldots, n$). The problem here is to evaluate the above probability and to find the optimal schedule of the starting times.

There are many realistic situations where the above policy describes the control of a process by an active redundant system. For example, we consider a data processing system with n available video displays that may work in parallel and where at least one operating display is sufficient. We assume that the system does not allow any human interference for inspection, unit replacement or repair during a required period c. Initially, a display starts its operation. To increase the system reliability, we schedule the starting times of the rest $(n - 1)$ displays. Thus, we optimize the system reliability by maximizing the reliability function with respect to the starting times s_i ($i = 2, \ldots, n$) and we follow that optimal schedule for the displays.

3. Model analysis

Let T be the system lifetime and T_i the independent unit-lifetime with pdf f_i and reliability function R_i ($i = 1, \ldots, n$), respectively. Let also $\mathscr{S}$ be the event of the successful control of the process during the required period of time c, i.e. $\mathscr{S} = \{T \geqslant c\}$ and $\mathscr{S}_i$ be the event of the successful control of the process up to time s_i, i.e. $\mathscr{S}_i = \{T \geqslant s_i\}$.

From the description of the policy above it follows that the system may be functioning with at most n units in parallel structure, even though the requirement for system operation is that a single unit is sufficient to operate. In this section we are interested in evaluating the system reliability $P[\mathscr{S}] = P[T \geqslant c]$ as a function of the starting times s_i ($i = 2, \ldots, n$). For notational convenience we put $s_{n+1} \equiv c$.

3.1. General case

Theorem 1. *The system reliability* $P[\mathscr{S}]$ *is given by*

$$P[\mathscr{S}] = P[\mathscr{S}_1]P[\mathscr{S}_2 \mid \mathscr{S}_1]P[\mathscr{S}_3 \mid \mathscr{S}_2]\cdots P[\mathscr{S}_{n+1} \mid \mathscr{S}_n],$$

where

$$\mathscr{S}_i = \{T \geqslant s_i\}, \ (i = 1, \ldots, n+1), \quad \mathscr{S} = \mathscr{S}_{n+1},$$

and

$$P[\mathscr{S}_{i+1}|\mathscr{S}_i] = r_i(s_{i+1}), \quad (i = 1, \ldots, n),$$

where

$$r_i(s) = \frac{r_{i-1}(s)}{r_{i-1}(s_i)} + \left\{1 - \frac{r_{i-1}(s)}{r_{i-1}(s_i)}\right\} R_i(s - s_i), \quad s > s_i, \quad (i = 2, \ldots, n),$$

with initial term

$$r_1(s) = R_1(s), \quad s \geqslant 0,$$

and $P[\mathscr{S}_1] = 1$.

Proof. The sequence of $\mathscr{S}_i$ $(i = 1, 2, \ldots)$, is decreasing and thus with $\mathscr{S}_{n+1} = \mathscr{S} = \{T \geqslant c\}$, we have:

$$\begin{aligned} P[\mathscr{S}] = P[\mathscr{S}_{n+1}] = P\left[\bigcap_{i=1}^{n+1} \mathscr{S}_i\right] &= P[\mathscr{S}_1]P[\mathscr{S}_2|\mathscr{S}_1]P[\mathscr{S}_3|\mathscr{S}_1\mathscr{S}_2]\cdots P[\mathscr{S}_{n+1}|\mathscr{S}_1\mathscr{S}_2\cdots\mathscr{S}_n] \\ &= P[\mathscr{S}_1]P[\mathscr{S}_2|\mathscr{S}_1]P[\mathscr{S}_3|\mathscr{S}_2]\cdots P[\mathscr{S}_{n+1}|\mathscr{S}_n]. \end{aligned} \tag{1}$$

Since $s_1 = 0$, the first term of the above product $P[\mathscr{S}_1]$, is

$$P[\mathscr{S}_1] = P[T \geqslant s_1] = P[T \geqslant 0] = 1,$$

while using the independence between T_1 and T_2, the second term $P[\mathscr{S}_2|\mathscr{S}_1]$, is

$$P[\mathscr{S}_2|\mathscr{S}_1] = P[T \geqslant s_2|T \geqslant s_1] = P[T \geqslant s_2] = R_1(s_2). \tag{2}$$

For the third term $P[\mathscr{S}_3|\mathscr{S}_2]$, we have:

$$\begin{aligned} P[\mathscr{S}_3|\mathscr{S}_2] &= P[T \geqslant s_3|T \geqslant s_2] = P[\max\{T_1, s_2 + T_2\} \geqslant s_3|T_1 \geqslant s_2] \\ &= 1 - P[T_1 < s_3, T_2 < s_3 - s_2|T_1 \geqslant s_2] = 1 - P[T_1 < s_3|T_1 \geqslant s_2]P[T_2 < s_3 - s_2|T_1 \geqslant s_2] \\ &= 1 - P[T_1 < s_3|T_1 \geqslant s_2]P[T_2 < s_3 - s_2] = 1 - \frac{P[s_2 \leqslant T_1 < s_3]}{P[T_1 \geqslant s_2]}\{1 - P[T_2 \geqslant s_3 - s_2]\} \\ &= 1 - \frac{R_1(s_2) - R_1(s_3)}{R_1(s_2)}\{1 - R_2(s_3 - s_2)\}, \end{aligned}$$

and thus

$$P[\mathscr{S}_3|\mathscr{S}_2] = \frac{R_1(s_3)}{R_1(s_2)} + \left\{1 - \frac{R_1(s_3)}{R_1(s_2)}\right\} R_2(s_3 - s_2). \tag{3}$$

Similarly, for the general term $P[\mathscr{S}_{i+1}|\mathscr{S}_i]$ $(i = 2, \ldots, n)$, we have:

$$\begin{aligned} P[\mathscr{S}_{i+1}|\mathscr{S}_i] &= P[T \geqslant s_{i+1}|T \geqslant s_i] = P[\max\{T_1, s_2 + T_2, \ldots, s_i + T_i\} \geqslant s_{i+1}|T_1 \geqslant s_2, \max\{T_1, s_2 + T_2\} \\ &\quad \geqslant s_3, \ldots, \max\{T_1, s_2 + T_2, \ldots, s_{i-1} + T_{i-1}\} \geqslant s_i] \\ &= P[\max\{\max\{T_1, s_2 + T_2, \ldots, s_{i-1} + T_{i-1}\}, s_i + T_i\} \geqslant s_{i+1}|T_1 \geqslant s_2, \max\{T_1, s_2 + T_2\} \\ &\quad \geqslant s_3, \ldots, \max\{T_1, s_2 + T_2, \ldots, s_{i-1} + T_{i-1}\} \geqslant s_i]. \end{aligned}$$

Let now

$$Y_i \equiv \max\{0, T_1, s_2 + T_2, \ldots, s_i + T_i\} - s_{i+1} \quad (i = 1, \ldots, n). \tag{4}$$

Using the above notation the event $\mathscr{S}_i$ can be written:

$$\begin{aligned}\mathscr{S}_i &= \{T_1 \geqslant s_2, \max\{T_1, s_2 + T_2\} \geqslant s_3, \ldots, \max\{T_1, s_2 + T_2, \ldots, s_{i-1} + T_{i-1}\} \geqslant s_i\} \\ &= \{Y_1 \geqslant 0, Y_2 \geqslant 0, \ldots, Y_{i-1} \geqslant 0\}\end{aligned} \tag{5}$$

and thus we have

$$\mathscr{S}_{i+1} = \mathscr{S}_i \cap \{Y_i \geqslant 0\} \quad (i = 1, \ldots, n).$$

Thus, the conditional probability $P[\mathscr{S}_{i+1}|\mathscr{S}_i]$ (i = 2,…,n), can be written:

$$\begin{aligned}P[\mathscr{S}_{i+1}|\mathscr{S}_i] &= P[\max\{s_i + Y_{i-1}, s_i + T_i\} \geqslant s_{i+1}|\mathscr{S}_{i-1}, Y_{i-1} \geqslant 0] \\ &= P[Y_{i-1} \geqslant 0, \max\{s_i + Y_{i-1}, s_i + T_i\} \geqslant s_{i+1}|\mathscr{S}_{i-1}]/P[Y_{i-1} \geqslant 0|\mathscr{S}_{i-1}] \\ &= \{1 - P[\{Y_{i-1} < 0\} \cup \{\max\{s_i + Y_{i-1}, s_i + T_i\} < s_{i+1}\}|\mathscr{S}_{i-1}]\}/P[Y_{i-1} \geqslant 0|\mathscr{S}_{i-1}] \\ &= \{1 - P[Y_{i-1} < 0|\mathscr{S}_{i-1}] - P[\max\{s_i + Y_{i-1}, s_i + T_i\} < s_{i+1}|\mathscr{S}_{i-1}] \\ &\quad + P[Y_{i-1} < 0, \max\{s_i + Y_{i-1}, s_i + T_i\} < s_{i+1}|\mathscr{S}_{i-1}]\}/P[Y_{i-1} \geqslant 0|\mathscr{S}_{i-1}],\end{aligned}$$

and thus,

$$\begin{aligned}P[\mathscr{S}_{i+1}|\mathscr{S}_i] &= \{P[Y_{i-1} \geqslant 0|\mathscr{S}_{i-1}] - P[\max\{s_i + Y_{i-1}, s_i + T_i\} < s_{i+1}|\mathscr{S}_{i-1}] \\ &\quad + P[Y_{i-1} < 0, \max\{s_i + Y_{i-1}, s_i + T_i\} < s_{i+1}|\mathscr{S}_{i-1}]\}/P[Y_{i-1} \geqslant 0|\mathscr{S}_{i-1}].\end{aligned} \tag{6}$$

Since $P[Y_{i-1} \geqslant 0|\mathscr{S}_{i-1}] = P[\mathscr{S}_i|\mathscr{S}_{i-1}]$, and due to the independence of T's, we can write:

$$\begin{aligned}P[\mathscr{S}_{i+1}|\mathscr{S}_i] &= \{P[\mathscr{S}_i|\mathscr{S}_{i-1}] - P[s_i + Y_{i-1} < s_{i+1}|\mathscr{S}_{i-1}]P[s_i + T_i < s_{i+1}|\mathscr{S}_{i-1}] \\ &\quad + P[Y_{i-1} < 0|\mathscr{S}_{i-1}]P[s_i + T_i < s_{i+1}|\mathscr{S}_{i-1}]\}/P[\mathscr{S}_i|\mathscr{S}_{i-1}] \\ &= 1 - \{P[s_i + Y_{i-1} < s_{i+1}|\mathscr{S}_{i-1}] - P[Y_{i-1} < 0|\mathscr{S}_{i-1}]\}P[T_i < s_{i+1} - s_i]/P[\mathscr{S}_i|\mathscr{S}_{i-1}],\end{aligned}$$

and thus

$$P[\mathscr{S}_{i+1}|\mathscr{S}_i] = 1 - P[s_i \leqslant s_i + Y_{i-1} < s_{i+1}|\mathscr{S}_{i-1}]\{1 - P[T_i \geqslant s_{i+1} - s_i]\}/P[\mathscr{S}_i|\mathscr{S}_{i-1}] \quad (i = 2, \ldots, n). \tag{7}$$

For notational convenience let us also define:

$$r_i(s) = P[T \geqslant s|T \geqslant s_i] \quad (s \geqslant 0) \ (i = 1, \ldots, n),$$

with

$$r_1(s) = R_1(s), \quad s \geqslant 0.$$

For $s = s_{i+1}$, we have:

$$r_i(s_{i+1}) = P[\mathscr{S}_{i+1} \mid \mathscr{S}_i],$$

and from (7)

$$r_i(s_{i+1}) = 1 - \frac{\{r_{i-1}(s_i) - r_{i-1}(s_{i+1})\}\{1 - R_i(s_{i+1} - s_i)\}}{r_{i-1}(s_i)} = \frac{r_{i-1}(s_{i+1})}{r_{i-1}(s_i)} + \left\{1 - \frac{r_{i-1}(s_{i+1})}{r_{i-1}(s_i)}\right\}R_i(s_{i+1} - s_i)$$
$$(i = 2, \ldots, n). \qquad \square$$

Applying the above recursive relation we get the final expression for the system reliability in closed form in terms of the unit's reliability functions R_i's ($i = 1,\ldots,n$).

4. Applications

In this section we give the final expressions for the system reliability applying Theorem 1 for $n = 3$ and 4. These expressions are used in Section 5 for exact analytic evaluations of the system reliability and the mean time to system failure (MTSF) with specific lifetime distributions. For larger systems, the application of the recursive expression in Theorem 1 becomes quite complicated. So, we have developed a program within the "*Mathematica*" environment which derives the final expression for the system reliability in closed form in terms of $R_i's$.

4.1. Three-unit parallel system

For $n = 3$ we have from (1)

$$P[\mathscr{S}] = P[\mathscr{S}_1]P[\mathscr{S}_2|\mathscr{S}_1]P[\mathscr{S}_3|\mathscr{S}_2]P[\mathscr{S}_4|\mathscr{S}_3], \tag{8}$$

with $P[\mathscr{S}_1] = 1$ and $P[\mathscr{S}_2|\mathscr{S}_1]$, $P[\mathscr{S}_3|\mathscr{S}_2]$ given by (2) and (3), respectively.

By Theorem 1 and with $s_4 = c$, the conditional probability $P[\mathscr{S}_4|\mathscr{S}_3]$ is:

$$P[\mathscr{S}_4|\mathscr{S}_3] = r_3(s_4), \tag{9}$$

where

$$r_3(s_4) = \frac{r_2(s_4)}{r_2(s_3)} + \left\{1 - \frac{r_2(s_4)}{r_2(s_3)}\right\} R_3(s_4 - s_3), \tag{10}$$

$$r_2(s_i) = \frac{r_1(s_i)}{r_1(s_2)} + \left\{1 - \frac{r_1(s_i)}{r_1(s_2)}\right\} R_2(s_i - s_2) \quad (i = 3, 4) \tag{11}$$

and

$$r_1(s_i) = R_1(s_i) \quad (i = 2, 3, 4). \tag{12}$$

Introducing the relations (10)–(12) into (9) we can write:

$$\begin{aligned} P[\mathscr{S}_4|\mathscr{S}_3] &= r_3(s_4) \\ &= \{R_1(s_4) + R_2(s_4 - s_2)[R_1(s_2) - R_1(s_4)] + \{R_1(s_3) + R_2(s_3 - s_2)[R_1(s_2) - R_1(s_3)] - R_1(s_4) \\ &\quad - R_2(s_4 - s_2)[R_1(s_2) - R_1(s_4)]\}R_3(s_4 - s_3)\}/\{R_1(s_3) + R_2(s_3 - s_2)[R_1(s_2) - R_1(s_3)]\}. \end{aligned} \tag{13}$$

Thus, from (2), (3) and (13), the relation (8) becomes:

$$\begin{aligned} P[\mathscr{S}] = {} & R_1(s_4) + R_2(s_4 - s_2)[R_1(s_2) - R_1(s_4)] + \{R_1(s_3) + R_2(s_3 - s_2)[R_1(s_2) - R_1(s_3)] - R_1(s_4) \\ & - R_2(s_4 - s_2)[R_1(s_2) - R_1(s_4)]\}R_3(s_4 - s_3). \end{aligned} \tag{14}$$

4.2. Four-unit parallel system

For $n = 4$ we have from (1)

$$P[\mathscr{S}] = P[\mathscr{S}_1]P[\mathscr{S}_2|\mathscr{S}_1]P[\mathscr{S}_3|\mathscr{S}_2]P[\mathscr{S}_4|\mathscr{S}_3]P[\mathscr{S}_5|\mathscr{S}_4]. \tag{15}$$

By Theorem 1 and with $s_5 = c$, the conditional probability $P[\mathscr{S}_5|\mathscr{S}_4]$ is:

$$P[\mathscr{S}_5|\mathscr{S}_4] = r_4(s_5), \tag{16}$$

where

$$r_4(s_5) = \frac{r_3(s_5)}{r_3(s_4)} + \left\{1 - \frac{r_3(s_5)}{r_3(s_4)}\right\} R_4(s_5 - s_4), \tag{17}$$

$$r_3(s_i) = \frac{r_2(s_i)}{r_2(s_3)} + \left\{1 - \frac{r_2(s_i)}{r_2(s_3)}\right\} R_3(s_i - s_3) \quad (i = 4, 5), \tag{18}$$

$$r_2(s_i) = \frac{r_1(s_i)}{r_1(s_2)} + \left\{1 - \frac{r_1(s_i)}{r_1(s_2)}\right\} R_2(s_i - s_2) \quad (i = 3, 4, 5) \tag{19}$$

and

$$r_1(s_i) = R_1(s_i) \quad (i = 2, 3, 4, 5). \tag{20}$$

Using relations (16)–(20) together with (14), relation (15) yields:

$$\begin{aligned} P[\mathscr{S}] = {} & \{R_1(s_5) + R_2(s_5 - s_2)R_1(s_2) - R_1(s_5)R_2(s_5 - s_2)\} \\ & \times \{1 - R_3(s_5 - s_3) - R_4(s_5 - s_4) + R_3(s_5 - s_3)R_4(s_5 - s_4)\} \\ & + \{R_1(s_4) + R_2(s_4 - s_2)R_1(s_2) - R_1(s_4)R_2(s_4 - s_2)\} \\ & \times \{R_4(s_5 - s_4) - R_3(s_4 - s_3)R_4(s_5 - s_4)\} \\ & + \{R_1(s_3) + R_2(s_3 - s_2)R_1(s_2) - R_1(s_3)R_2(s_3 - s_2)\} \\ & \times \{R_3(s_5 - s_3) - R_3(s_5 - s_3)R_4(s_5 - s_4) + R_3(s_4 - s_3)R_4(s_5 - s_4)\}. \end{aligned} \tag{21}$$

5. Analytical results with special distributions

In this section we present analytical results for the reliability function of a three and four-unit system with independent unit-lifetimes which are either identically or not identically Weibull distributed. Optimal starting times and the corresponding MTSF are evaluated. Comparisons between the policy applied here (Policy I), and the policy applied in Papageorgiou and Kokolakis [6] (Policy II), are made.

5.1. Reliability of an identical three-unit system with Weibull lifetimes

In the case of a three-unit system with independent and identically distributed unit-lifetimes following a Weibull distribution with pdf $f(t) = ab^{-a}e^{-(\frac{t}{b})^a}t^{a-1}$, and with $s_1 = 0$, $s_4 = c$, the reliability of the system $P[\mathscr{S}]$ from (14), is given by

$$\begin{aligned} P[\mathscr{S}] &= P[T \geqslant c] \\ &= e^{-b^{-a}c^a} - e^{-b^{-a}c^a - b^{-a}(c-s_2)^a} + e^{-b^{-a}(c-s_2)^a - b^{-a}s_2^a} - e^{-b^{-a}c^a - b^{-a}(c-s_3)^a} + e^{-b^{-a}c^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_3)^a} \\ &\quad - e^{-b^{-a}s_2^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_3)^a} + e^{-b^{-a}s_3^a - b^{-a}(c-s_3)^a} + e^{-b^{-a}s_2^a - b^{-a}(c-s_3)^a - b^{-a}(s_3-s_2)^a} \\ &\quad - e^{-b^{-a}s_3^a - b^{-a}(c-s_3)^a - b^{-a}(s_3-s_2)^a}. \end{aligned} \tag{22}$$

If the lifetimes of the three units follow a special Weibull distribution with $a = 2$ and $b = 1$, i.e. when $f(t) = 2te^{-t^2}$, then from (22) we have:

$$\begin{aligned} P[\mathscr{S}] &= P[T \geqslant c] \\ &= e^{-c^2} - e^{-c^2-(c-s_2)^2} + e^{-s_2^2-(c-s_2)^2} - e^{-c^2-(c-s_3)^2} + e^{-s_3^2-(c-s_3)^2} + e^{-c^2-(c-s_2)^2-(c-s_3)^2} \\ &\quad - e^{-s_2^2-(c-s_2)^2-(c-s_3)^2} + e^{-s_2^2-(c-s_3)^2-(s_3-s_2)^2} - e^{-s_3^2-(c-s_3)^2-(s_3-s_2)^2}. \end{aligned} \tag{23}$$

In this case the mean lifetime of the units is $E[T] = \frac{\sqrt{\pi}}{2}$. The reliability function (23), for several starting times s_2, s_3 and $c = 1.5$ is shown in Fig. 2. After optimizing the above expression with respect to s_2 and s_3 (Policy I), we get $s_2 = 0.364$, $s_3 = 0.898$ and $P[\mathscr{S}] = 0.632$. The corresponding mean time to system failure in this case is MTSF = 1.48.

In Papageorgiou and Kokolakis [6], an n-unit standby redundant system was studied. Two units start their operation simultaneously at time $s = 0$ and one of these is replaced by a new one upon its failure (Policy II). The authors showed that when the lifetimes of the three units are independent and follow a Weibull distribution with the same parameters $a = 2$ and $b = 1$, the system reliability is given by

$$\begin{aligned} P[\mathscr{S}] &= P[T \geqslant c] \\ &= -\frac{5}{3}e^{-2c^2} + \frac{8}{3}e^{-c^2} - ce^{-\frac{3}{2}c^2}\sqrt{2\pi}\,\mathrm{Erf}\left[\frac{c}{\sqrt{2}}\right] + \frac{2}{3}ce^{-\frac{2}{3}c^2}\sqrt{\frac{\pi}{3}}\mathrm{Erf}\left[\frac{c}{\sqrt{3}}\right] + \frac{2}{3}ce^{-\frac{2}{3}c^2}\sqrt{\frac{\pi}{3}}\mathrm{Erf}\left[\frac{2c}{\sqrt{3}}\right], \end{aligned} \tag{24}$$

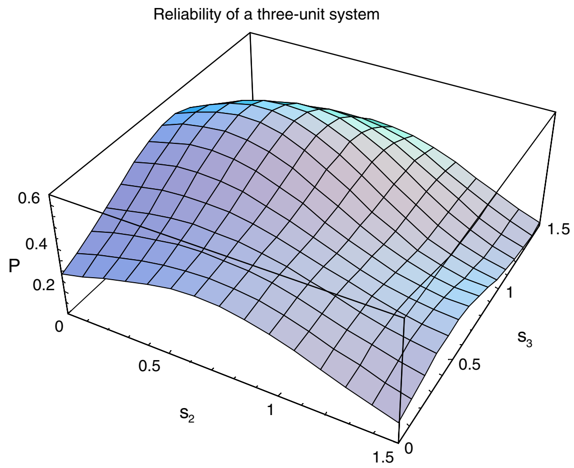

Fig. 2. System reliability as a function of starting times s_2, s_3, for required period $c = 1.5$ and with parameters $\alpha = 2$ and $b = 1$.

Table 1
Weibull iid ($n = 3$, $a = 2$, $b = 1$)

Reliability of the systems		
c	Policy I	Policy II
0.3	0.9996	0.9998
0.6	0.9870	0.9912
0.9	0.9266	0.9330
1.2	0.8011	0.7799
1.5	0.6320	0.5541
1.8	0.4563	0.3322

where

$$\mathrm{Erf}[z] = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2} \mathrm{d}t.$$

In Table 1, the system reliability is presented for the above Policies I and II and for several values of c. It is apparent that Policy II is superior at high reliability levels, while at low reliability levels Policy I is superior.

5.2. Reliability of an identical four-unit system with Weibull lifetimes

In the case of a four-unit system with independent unit-lifetimes following a Weibull distribution with parameters a and b, with $s_1 = 0$ and $s_5 = c$, the reliability of the system $P[\mathscr{S}]$ from (21), is given by

$$\begin{aligned}
P[\mathscr{S}] &= P[T \geqslant c] \\
&= \mathrm{e}^{-b^{-a}c^a} - \mathrm{e}^{-b^{-a}c^a - b^{-a}(c-s_2)^a} + \mathrm{e}^{-b^{-a}(c-s_2)^a - b^{-a}s_2^a} - \mathrm{e}^{-b^{-a}c^a - b^{-a}(c-s_3)^a} + \mathrm{e}^{-b^{-a}c^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_3)^a} \\
&\quad - \mathrm{e}^{-b^{-a}s_2^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_3)^a} + \mathrm{e}^{-b^{-a}s_3^a - b^{-a}(c-s_3)^a} + \mathrm{e}^{-b^{-a}s_2^a - b^{-a}(c-s_3)^a - b^{-a}(s_3-s_2)^a} \\
&\quad - \mathrm{e}^{-b^{-a}s_3^a - b^{-a}(c-s_3)^a - b^{-a}(s_3-s_2)^a} - \mathrm{e}^{-b^{-a}c^a - b^{-a}(c-s_4)^a} + \mathrm{e}^{-b^{-a}c^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_4)^a} \\
&\quad - \mathrm{e}^{-b^{-a}s_2^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_4)^a} + \mathrm{e}^{-b^{-a}c^a - b^{-a}(c-s_3)^a - b^{-a}(c-s_4)^a} - \mathrm{e}^{-b^{-a}c^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_3)^a - b^{-a}(c-s_4)^a} \\
&\quad + \mathrm{e}^{-b^{-a}s_2^a - b^{-a}(c-s_2)^a - b^{-a}(c-s_3)^a - b^{-a}(c-s_4)^a} - \mathrm{e}^{-b^{-a}s_3^a - b^{-a}(c-s_3)^a - b^{-a}(c-s_4)^a}
\end{aligned}$$

$$
\begin{aligned}
&- e^{-b^{-a}s_2^a-b^{-a}(c-s_3)^a-b^{-a}(s_3-s_2)^a-b^{-a}(c-s_4)^a} + e^{-b^{-a}s_3^a-b^{-a}(c-s_3)^a-b^{-a}(s_3-s_2)^a-b^{-a}(c-s_4)^a} + e^{-b^{-a}s_4^a-b^{-a}(c-s_4)^a} \\
&+ e^{-b^{-a}s_2^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_2)^a} - e^{-b^{-a}s_4^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_2)^a} + e^{-b^{-a}s_3^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_3)^a} \\
&+ e^{-b^{-a}s_2^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_3)^a-b^{-a}(s_3-s_2)^a} - e^{-b^{-a}s_3^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_3)^a-b^{-a}(s_3-s_2)^a} \\
&- e^{-b^{-a}s_4^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_3)^a} - e^{-b^{-a}s_2^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_2)^a-b^{-a}(s_4-s_3)^a} + e^{-b^{-a}s_4^a-b^{-a}(c-s_4)^a-b^{-a}(s_4-s_2)^a-b^{-a}(s_4-s_3)^a}.
\end{aligned}
\tag{25}
$$

If the lifetimes of the three units follow a special Weibull distribution with $a=2$ and $b=1$, i.e. when $f(t)=2te^{-t^2}$, then from (25) we have:

$$
\begin{aligned}
P[\mathscr{S}] &= P[T \geqslant c] \\
&= e^{-c^2} - e^{-c^2-(c-s_2)^2} + e^{-s_2^2-(c-s_2)^2} - e^{-c^2-(c-s_3)^2} + e^{-s_3^2-(c-s_3)^2} + e^{-c^2-(c-s_2)^2-(c-s_3)^2} \\
&\quad - e^{-s_2^2-(c-s_2)^2-(c-s_3)^2} + e^{-s_2^2-(c-s_3)^2-(s_3-s_2)^2} - e^{-s_3^2-(c-s_3)^2-(s_3-s_2)^2} - e^{-c^2-(c-s_4)^2} + e^{-c^2-(c-s_2)^2-(c-s_4)^2} \\
&\quad - e^{-s_2^2-(c-s_2)^2-(c-s_4)^2} + e^{-c^2-(c-s_3)^2-(c-s_4)^2} - e^{-c^2-(c-s_2)^2-(c-s_3)^2-(c-s_4)^2} + e^{-s_2^2-(c-s_2)^2-(c-s_3)^2-(c-s_4)^2} \\
&\quad - e^{-s_3^2-(c-s_3)^2-(c-s_4)^2} - e^{-s_2^2-(c-s_3)^2-(c-s_4)^2-(s_3-s_2)^2} + e^{-s_3^2-(c-s_3)^2-(c-s_4)^2-(s_3-s_2)^2} + e^{-s_4^2-(c-s_4)^2} \\
&\quad + e^{-s_2^2-(c-s_4)^2-(s_4-s_2)^2} - e^{-s_4^2-(c-s_4)^2-(s_4-s_2)^2} + e^{-s_3^2-(c-s_4)^2-(s_4-s_3)^2} + e^{-s_2^2-(c-s_4)^2-(s_4-s_3)^2-(s_3-s_2)^2} \\
&\quad - e^{-s_4^2-(c-s_4)^2-(s_4-s_3)^2} - e^{-s_3^2-(c-s_4)^2-(s_4-s_3)^2-(s_3-s_2)^2} - e^{-s_2^2-(c-s_4)^2-(s_4-s_3)^2-(s_4-s_2)^2} \\
&\quad + e^{-s_4^2-(c-s_4)^2-(s_4-s_3)^2-(s_4-s_2)^2}.
\end{aligned}
\tag{26}
$$

Here, again, the mean lifetime of the units is $E[T]=\frac{\sqrt{\pi}}{2}$. After optimizing the above expression with respect to s_2, s_3 and s_4, for $c=1.5$, we get $s_2=0.184$, $s_3=0.607$ and $s_4=1.016$ and $P[\mathscr{S}]=0.814$. The corresponding mean time to system failure in this case is MTSF = 1.82.

5.3. Reliability of a non-identical three-unit system with Weibull lifetimes

In the case of a three-unit system with independent unit-lifetimes following Weibull distributions with different parameters (a_i, b_i), for $i=1$, 2 and 3, then from (14), with $s_1=0$ and $s_4=c$, the reliability of the system $P[\mathscr{S}]$ is given by

$$
\begin{aligned}
P[\mathscr{S}] &= P[T \geqslant c] \\
&= e^{-b_1^{-a_1}c^{a_1}} - e^{-b_1^{-a_1}c^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}} + e^{-b_2^{-a_2}(c-s_2)^{a_2}-b_1^{-a_1}s_2^{a_1}} - e^{-b_1^{-a_1}c^{a_1}-b_3^{-a_3}(c-s_3)^{a_3}} \\
&\quad + e^{-b_1^{-a_1}c^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}} - e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}} + e^{-b_1^{-a_1}s_3^{a_1}-b_3^{-a_3}(c-s_3)^{a_3}} \\
&\quad + e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}} - e^{-b_1^{-a_1}s_3^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}}.
\end{aligned}
\tag{27}
$$

If the lifetimes of the three units follow special Weibull distributions with $(a_1,b_1)=(2,1)$, i.e. $f_1(t)=2te^{-t^2}$, $(a_2,b_2)=(2,1.5)$, i.e. $f_2(t)=0.888889te^{-0.444444t^2}$ and $(a_3,b_3)=(2,2)$, i.e. $f_3(t)=\frac{1}{2}te^{\frac{-t^2}{4}}$, then from (27) we get an analytic expression for the system reliability. In this case the units mean lifetimes are $E[T_1]=0.8862$, $E[T_2]=1.3293$ and $E[T_3]=1.7725$. After optimizing the derived expression for $c=1.5$ with respect to s_2 and s_3, we get $s_2=0.113$, $s_3=0.615$ and $P[\mathscr{S}]=0.870$. The corresponding mean time to system failure in this case is MTSF = 2.42.

5.4. Reliability of a non-identical four-unit system with Weibull lifetimes

In the case of a four-unit system with independent unit-lifetimes following Weibull distributions with different parameters (a_i, b_i), for $i=1,2,3$ and 4, then from (21), with $s_1=0$ and $s_5=c$, the reliability of the system $P[\mathscr{S}]$ is given by

$$
\begin{aligned}
P[\mathscr{S}] &= P[T \geqslant c] \\
&= e^{-b_1^{-a_1}c^{a_1}} - e^{-b_1^{-a_1}c^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}} + e^{-b_2^{-a_2}(c-s_2)^{a_2}-b_1^{-a_1}s_2^{a_1}} - e^{-b_1^{-a_1}c^{a_1}-b_3^{-a_3}(c-s_3)^{a_3}} \\
&\quad + e^{-b_1^{-a_1}c^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}} - e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}} + e^{-b_1^{-a_1}s_3^{a_1}-b_3^{-a_3}(c-s_3)^{a_3}} \\
&\quad + e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}} - e^{-b_1^{-a_1}s_3^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}} - e^{-b_1^{-a_1}c^{a_1}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad + e^{-b_1^{-a_1}c^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_4^{-a_4}(c-s_4)^{a_4}} - e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad + e^{-b_1^{-a_1}c^{a_1}-b_3^{-a_3}(c-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} - e^{-b_1^{-a_1}c^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad + e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(c-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} - e^{-b_1^{-a_1}s_3^{a_1}-b_3^{-a_3}(c-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad - e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} + e^{-b_1^{-a_1}s_3^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(c-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad + e^{-b_1^{-a_1}s_4^{a_1}-b_4^{-a_4}(c-s_4)^{a_4}} + e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(s_4-s_2)^{a_2}-b_4^{-a_4}(c-s_4)^{a_4}} - e^{-b_1^{-a_1}s_4^{a_1}-b_2^{-a_2}(s_4-s_2)^{a_2}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad + e^{-b_1^{-a_1}s_3^{a_1}-b_3^{-a_3}(s_4-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} + e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(s_4-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad - e^{-b_1^{-a_1}s_3^{a_1}-b_2^{-a_2}(s_3-s_2)^{a_2}-b_3^{-a_3}(s_4-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} - e^{-b_1^{-a_1}s_4^{a_1}-b_3^{-a_3}(s_4-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} \\
&\quad - e^{-b_1^{-a_1}s_2^{a_1}-b_2^{-a_2}(s_4-s_2)^{a_2}-b_3^{-a_3}(s_4-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}} + e^{-b_1^{-a_1}s_4^{a_1}-b_2^{-a_2}(s_4-s_2)^{a_2}-b_3^{-a_3}(s_4-s_3)^{a_3}-b_4^{-a_4}(c-s_4)^{a_4}}. \qquad (28)
\end{aligned}
$$

If the lifetimes of the three units follow special Weibull distributions where (a_i, b_i) $(i = 1, \ldots, 3)$ are same as in case of the previous section, and $(a_4, b_4) = (2, 2.5)$ i.e. $f_4(t) = 0.32te^{-0.16t^2}$ and $E[T_4] = 2.2156$, then from (28) we get an analytic expression for the system reliability. After optimizing the derived expression with respect to s_2, s_3 and s_4, for $c = 1.5$, we get $s_2 = 0.022$, $s_3 = 0.302$ and $s_4 = 0.753$ and $P[\mathscr{S}] = 0.978$. The corresponding mean time to system failure in this case is MTSF = 3.22.

Remark. In the case of different lifetime distributions it might be of interest to optimize the system with respect to the order the units are introduced. The optimal ordering of the units can be easily specified by considering all possible permutations of the unit reliability functions $R_i's$ in the derived from Theorem 1 final closed form of the system reliability. In the case of four non-identical units with Weibull lifetimes we may conclude from the final closed form (28) that the probability of successful control is maximized by using the best, i.e. the one with the largest mean lifetime, first. For example, with $c = 1.5$, $a_i = 2$ $(i = 1, \ldots, 4)$, $b_1 = 2.5$, $b_2 = 2$, $b_3 = 1.5$ and $b_4 = 1$ and s_i $(i = 1, \ldots, 4)$ the corresponding optimal starting times, we obtain $P[\mathscr{S}] = 0.987$, whereas with $b_1 = 1$, $b_2 = 1.5$, $b_3 = 2$ and $b_4 = 2.5$ we get, as mentioned before, $P[\mathscr{S}] = 0.978$. For the same parameters, and with $c = 2.5$, we also obtain maximum probability $P[\mathscr{S}] = 0.872$, for descending $b_i's$. We reach the same conclusion in the case of Exponential unit-lifetimes.

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